# AN INVITATION TO MODULAR FORMS

*by*

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***Abstract.*** — In these lecture notes we give a short introduction of the theory of modular forms for $\mathrm{SL}(2,\mathbb{Z})$. We will stick to the geometric point of view (and let arithmetic aspects aside).

The only required background is a general course of complex analysis. In particular we will not assume the knowledge of Riemann surfaces (nor of manifold actually), but it shall certainly help to understand the definitions. Besides, we emphasize that some important notions will be developed in the exercises.

Good references to go further this course are the book of J.P. Serre ([**Ser73**]), the online course notes of J.S. Milne ([**J.S15**]) or the "1-2-3 of Modular forms" by Bruinier, Van der Geer, Harder, and Zagier ( [**BvdGHZ08**]).

## Contents



## 1. The Poincaré half-plane and its $\mathrm{SL}(2,\mathbb{Z})$ action

**1.1. Action of $\mathrm{SL}(2,\mathbb{C})$.** — If $R=\mathbb{Z},\mathbb{R}$, or $\mathbb{C}$ we denote by $\mathrm{SL}(2,R)$ the *special linear group for $R$*

$$\left\{\gamma=\begin{pmatrix} a & b\\ c & d\end{pmatrix}, (a,b,c,d)\in R^4 | \det(\gamma)=ad-bc=1\right\}.$$

(with the matrix multiplication).

***Remark 1.1.*** — To check that this is a group (for the multiplication), what we use is the fact that $\det(\gamma_1\cdot\gamma_2)=\det(\gamma_1)\det(\gamma_2)$ (thus the multiplication is well-defined). Besides, the inverse of $\gamma$ is in $\mathrm{SL}(2,R)$ as proved in linear algebra courses if $R$ is a field and simply by direct computation if $R=\mathbb{Z}$, i.e. $\begin{pmatrix} a & b\\ c & d\end{pmatrix}^{-1}=\begin{pmatrix} d & -b\\ -c & a\end{pmatrix}$.

The group $\{\mathrm{Id}, -\mathrm{Id}\}$ is a normal subgroup of $\mathrm{SL}(2,R)$. We denote by $\mathrm{PSL}(2,R)$ the associated quotient.

As a large part of the text will be devoted to the study of $\mathrm{SL}(2,\mathbb{Z})$, we should recall a presentation of this group.

***Proposition 1.2.*** — *The group* $\mathrm{SL}(2,\mathbb{Z})$ *is generated by the two elements* $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ *and* $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. *Besides it can be presented as* $\langle S,T\rangle/(S^4,(ST)^3S^{-2})$ *and* $\mathrm{PSL}(2,\mathbb{Z}) = \langle S,T\rangle/(S^2,(ST)^3)$.

The proof will be given in Exercise 1.5.

**1.2. Automorphism groups of complex domains.** — We will consider the set $\mathbb{P} = \mathbb{C} \sqcup \infty$. This set has a natural topology that makes it isomorphic to the 2-sphere (think of the infinity as the north pole and of 0 as the south pole). We say that an endomorphism $f$ of $\mathbb{P}$ is *holomorphic* if its restriction to $\mathbb{C} \cap f^{-1}(\mathbb{C})$ is an holomorphic function. We denote by $\mathrm{Aut}(\mathbb{P})$ the set of *holomorphic automorphisms* (we will often simply call them automorphisms).

For any open domain $U \subset \mathbb{C}$ we denote by $\mathrm{Aut}(U)$ the group of holomorphic bijections of $U$.

***Proposition 1.3.*** — *The group* $\mathrm{Aut}(\mathbb{P})$ *is isomorphic to* $\mathrm{PSL}(2,\mathbb{C})$. *The isomorphism is defined by: for all* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C})$, *we denote*

$$\gamma(z) = \begin{cases} \frac{az+b}{cz+d}, & \text{if } z \in \mathbb{C} \setminus \{-d/c\} \\ \infty, & \text{if } z = -d/c \\ a/c, & \text{if } z = \infty. \end{cases}$$

*The group of automorphism of* $\mathbb{C}$ *is the group of affine transformations.*

*Proof.* — The above definition gives an action of $\mathrm{SL}(2,\mathbb{C})$ on $\mathbb{P}$ that descends to an action of $\mathrm{PSL}(2,\mathbb{C})$ as $-\mathrm{Id}(z) = z$. Besides, it is obvious that $\gamma$ is holomorphic as defined above. Thus $\mathrm{PSL}(2,\mathbb{C})$ is a subgroup of $\mathrm{Aut}(\mathbb{P})$. To check that the converse is also true, we just need to remark that composing any automorphism $f$ of $\mathbb{P}$ with $z \mapsto 1/(z - f(\infty))$ we get an automorphism of $\mathbb{C}$ so the statement it falls from the second part of the proposition.

Now let $f$ be an automorphism of $\mathbb{C}$, that we write $f(z) = \sum_{n\geq 0} a_n z^n$. Then there are two possibilities:

- Either a finite number of the $a_n$ are non zeros. In which case, $f$ is a polynomial (of degree $d$) and we know that a generic point of $\mathbb{C}$ has $d$ preimages under $f$, thus $d$ is an affine transformation (a polynomial of degree 1).
- If an infinite number of $a_n$'s are non zero then the function $z \mapsto f(1/z)$ has an essential singularity at 0. By the Casorati-Weirstrass theorem, any neighboorhod of 0 has a dense preimage in $\mathbb{C}$. This contradicts the injectivity of $f$.

$\square$

**1.3. Poincaré half-plane.** — The *Poincaré half-plane* $\mathbb{H} \subset \mathbb{C}$ is the set of complex numbers with strictly positive values. This set is isomorphic to the open disk $\Delta_1$ via the map: $z \mapsto \frac{z-i}{z+i}$ (we will denote by $\Delta_r \subset \mathbb{C}$ the disk of radius $r$ in $\mathbb{C}$ and center 0).

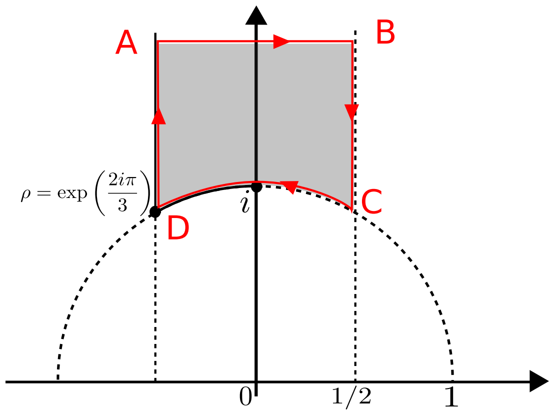


FIGURE 1. In grey, the Dirichlet domain

***Proposition 1.4.*** — *The automorphism group of $\mathbb{H}$ is the group $\mathrm{PSL}(2,\mathbb{R})$ (acting by homography as in the case of the plane). Equivalently the set of isomorphisms of $\Delta_1$ is the set of functions $z \mapsto \beta \cdot \frac{z-\alpha}{1-z\bar{\alpha}}$ where $\beta$ and $\alpha$ are complex numbers of norm respectively equal and less than 1.*

*Proof.* — We can easily check that $\gamma(z) \in \mathbb{H}$ for all $\gamma \in \mathrm{SL}(2,\mathbb{R})$ and $z \in \mathbb{H}$. Now to prove that the automorphism group $\mathrm{PSL}(2,\mathbb{R})$ is the full automorphism group, we shall work with the disk. Let $f$ be an automorphism of $\Delta_1$. We can still assume that we assume that $f(0) = 0$. Then by the maximum principle(see Exercise 2.3) we can check that $1 \leq |f(z)/z| \leq 1$ thus the maximum principle implies that $f(z)/z$ is a constant (of norm 1). □

In the text we will be mainly interested in the action of $\mathrm{SL}(2,\mathbb{Z}) \subset \mathrm{SL}(2,\mathbb{R})$ on $\mathbb{H}$. Let us begin by describing the quotient of this action.

A *fundamental domain* for the action of $\mathrm{SL}(2,\mathbb{Z})$ on $\mathbb{H}$ is an open set $D \subset \mathbb{H}$ such that:

- $\gamma(z) = z$ implies $\gamma = \pm\mathrm{Id}$ for all $z \in D$ and $\gamma \in \mathrm{SL}(2,\mathbb{Z})$;
- for all $z \in \mathbb{H}$, there exists $\gamma \in \mathrm{SL}(2,\mathbb{Z})$ and $z_0 \in \overline{D}$ such that $\gamma z_0 = z$.

The classical fundamental domain for the action $\mathrm{SL}(2,\mathbb{Z})$ on $\mathbb{H}$ is the open set (the *Dirichlet domain*)

$$D_{Dir} = \{z \in \mathbb{H} \ / \ |z| > 1, |\Re(z)| < 1/2\}.$$

See Exercise 1.6 for a proof. To understand what happens on the boundary of this domain, we just need to see that the element $T$ maps the line $\Re(z) = -1/2$ to the line $\Re(z) = -1/2$ and that $S$ fixes the unit circle and acts as an axial symmetry. Thus the quotient $\mathbb{H}/SL(2,\mathbb{Z})$ (as a set) is natural bijection with

$$\overline{D_{Dir}} \setminus (\{z/|z| = 1, \Re(z) > 0\} \cup \{z/\Re(z) = 1/2\})$$

(see figure 1). Besides this quotient has two fixed points under the action of $PSL(2,\mathbb{Z})$ the complex numbers $i$ (fixed under $S$) and $\rho = \exp(2i\pi/3)$ (fixed under $ST$).

### 1.4. Exercises. —

***Exercise 1.5.*** — (1) Using the fact that

$$S \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -c & -d \\ a & b \end{pmatrix} \text{ and } T^n \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+cn & b+nd \\ c & d \end{pmatrix}$$

show that $S$ and $T$ generates $\mathrm{SL}(2,\mathbb{Z})$. (tip: begin by showing that we can get rid of the coefficient $c$).

(2) Now consider the group $\mathrm{SL}(2,\mathbb{Z})$ and consider the element $B = TS$. A reduced word on $S$ and $B$ is a sequence of elements $(A_i)_{1\leq i\leq n} \in \{S, B, B^{-1}\}$ such that $A_i \in \{B, B^{-1}\} \Leftrightarrow A_{i+1} = S$ for all $1 \leq i < n$. Show that $A_1 \dots A_n$ is not the identity (tip: consider $P = \mathbb{R}_+ \cap (\mathbb{R} \setminus \mathbb{Q})$ and $N = \mathbb{R}_- \cap (\mathbb{R} \setminus \mathbb{Q})$ and use the fact that $S(P) \subset N$ while $B(N)$ and $B^{-1}(N) \subset (P)$).

***Exercise 1.6.*** — (1) Let $z$ in $\mathbb{H}$. Show that there exists a $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$ such that $|cz + d|$ is minimal. Show that the imaginary part of $\gamma(z)$ is maximal.

(2) Deduce from this that there exists a $\gamma'$ such that $\gamma' \cdot z \in \overline{D_{Dir}}$. Conclude that the $D_{Dir}$ is a fundamental domain for the action of $\mathrm{SL}(2,\mathbb{Z})$.

## 2. Modular forms

We give here a first definition of modular forms that unfortunately will not allow to construct any interesting examples of such objects.

**2.1. First definition.** — From now on, we generally denote by $\tau$ the coordinates of $\mathbb{H}$ to distinguish it from the coordinate $z$ of the plane $\mathbb{C}$.

If $f$ is a function on $\mathbb{H}$ and $\gamma \in \mathrm{SL}(2,\mathbb{R})$, we define $\gamma[f]$ by $\gamma[f](\tau) = f(\gamma(\tau))$.

***Definition 2.1.*** — Let $k \in \mathbb{Z}$. An holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is *weakly modular* of weight $k$ if for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$ we have

$$\gamma[f](\tau) = f(\tau) \cdot (c\tau + d)^k$$

for all $\tau \in \mathbb{H}$.

A function $f$ is a *modular form of weight* $k$ if it is weakly modular of weight $k$ and for all $x \in \mathbb{R}$, the function $f(iy + x)$ converges to a unique value $f(\infty)$ as $y \to \infty$ with real values.

In general, a *modular form* is a linear combination of modular forms of modular forms of given weights.

For all $k \in \mathbb{Z}$, we denote by $M_k$ the space of modular forms of weight $k$ and by $M_* = \bigoplus_k M_k$ the space of modular forms.

**2.2. Remarks.** — Let us give some important remarks:

- Using the action of $-\mathrm{Id}$ we can see that $(-\mathrm{Id})[f] = (-1)^k f$. Thus any modular form of odd weight is null.
- Following this train of idea we can check that $S[f](i) = i^k f(i)$ and $(ST)[f](\rho) = \rho^k f(\rho)$ for any modular form of weight $k$. Thus $f(i) = 0$ if $k \not\equiv 0[4]$ and $f(\rho) = 0$ if $k \not\equiv 0[3]$.
- constant functions are quasi-modular of weight 0.
- if $f$ and $f'$ are modular of weight $k$ and $k'$ then $ff'$ is modular of weight $k + k'$.

Besides we can use the maximum principle.

***Lemma 2.2.*** — *Let $f$ be a modular form that does not vanish on $\mathbb{H}$ or $\infty$ then $f$ is a constant.*

*Proof.* — See Exercise 2.3 □

**2.3. The $q$-expansion.** — Let us denote by $\mathbb{H}_{>1} = \{\tau/\Im(\tau) > 1\}$. The following function will be of great importance:

$$\begin{array}{rcl} q : \mathbb{H}_{>1} & \to & \Delta_{\exp(-2\pi)} \\ \tau & \mapsto & \exp(2i\pi\tau) \end{array}$$

Let us remark that $q(\tau) = q(\tau')$ if and only if $\tau = \tau' + a$ with $a \in \mathbb{Z}$. In particular if a function $f$ on $\mathbb{H}_{>1}$ is invariant under $T$ then it defines a function $\widetilde{f}$ on $\Delta^*_{\exp(-2\pi)} = \Delta_{\exp(-2\pi)} \setminus \{0\}$, such that $f = \widetilde{f} \circ q$. By abuse of notation we will often denote by $f$ the function $\widetilde{f}$.

Now, remark that imposing that $f$ has a limit when $\tau$ goes to infinity, is equivalent to asking that $f$ extends to an holomorphic function on $\Delta_{\exp(-2\pi)}$. In particular if $f$ is modular form we can write $f(q) = \sum_{n\geq 0} a_n q^n$, this is called the *$q$-expansion of $f$*.

**2.4. Exercises.** —

***Exercise 2.3.*** — We recall that the maximum principle is the following theorem in complex analysis: *Let $f$ be an holomorphic function defined on $\Delta_r$ and let $0 < r' < r$. If the maximum (or minimum) of $f$ on $\overline{\Delta}_{r'}$ is reached at some point of $\overline{\Delta}_{r'} \setminus \Delta_{r'}$ then $f$ is constant.* Use this theorem to show that any modular form of weight 0 is a constant.

## 3. Moduli space of lattices

The purpose of the present section is to construct the first non-trivial modular forms, the Eisenstein series.

**3.1. Lattices.** — A *lattice* $\Lambda$ of $\mathbb{C}$ is a discrete subgroup for the sum (here discrete means that there exists $\epsilon > 0$ such that $g \in \Lambda \cap \Delta_\epsilon$ implies $g = 0$). One can easily check that all lattices of $\mathbb{C}$ can be written as $\Lambda = \omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z}$ where $\omega_1$ and $\omega_2$ are $\mathbb{R}$-linearly independent.

We say that two lattices $\Lambda$ and $\Lambda'$ are *equivalent* if there exists $\lambda \in \mathbb{C}$ such that $\lambda\Lambda = \Lambda'$.

***Proposition 3.1.*** — *Any lattice is equivalent to a lattice $\Lambda(\tau) = \mathbb{Z} \oplus \tau\mathbb{Z}$ for some $\tau \in \mathbb{H}$. Besides, if $\tau$ and $\tau'$ are in $\mathbb{H}$ then $\Lambda(\tau)$ is equivalent to $\Lambda(\tau')$ if and only if $\tau = \gamma(\tau')$ for some $\gamma \in \mathrm{SL}(2,\mathbb{Z})$.*

*Proof.* — The lattice $\omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z}$ is isomorphic to $\mathbb{Z} \oplus \frac{\omega_1}{\omega_2}\mathbb{Z}$ and to $\mathbb{Z} \oplus \frac{\omega_2}{\omega_1}\mathbb{Z}$ thus any lattice is isomorphic to a lattice $\Lambda(\tau) = \mathbb{Z} \oplus \tau\mathbb{Z}$ for some $\tau \in \mathbb{H}$.

Now let us fix $\tau$ and $\tau'$ in $\mathbb{H}$ such that $\Lambda(\tau)$ is equivalent to $\Lambda(\tau')$. Then there exists $\lambda \in \mathbb{C}$, and coordinates $a, b, c, d$ in $\mathbb{Z}$ such that $\tau = \lambda(a\tau' + b)$ and $1 = \lambda(c\tau' + d)$, thus $\tau = \frac{a\tau'+b}{c\tau'+d}$. To conclude that $ad - bc = 1$, it suffices to see that the matrix $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ defines a $\mathbb{Z}$-linear isomorphisms between the lattices $\Lambda(\tau')$ and $\Lambda(\tau)$. The general theory of $\mathbb{Z}$-module implies that this matrix is $\mathbb{Z}$-invertible and thus its determinant is $\pm 1$. The determinant is 1 as this linear isomorphism preserves the orientation of the plane. □

In other words the set $\mathbb{H}/\mathrm{SL}(2,\mathbb{Z})$ is the set of isomorphism classes (or the moduli space of) lattices of $\mathbb{C}$.

**3.2. Eisenstein series.** — In order to define modular forms we will try the naive approach of cooking functions on the space of lattices. The *Eisenstein series* of weight $k$ is the function

$$G_k(\Lambda) = \sum_{\omega \in \Lambda^*} \frac{1}{\omega^k}.$$

If we fix a lattice $\Lambda$, we can check that this series is absolutely convergent if $k \geq 3$ (in fact if $k \geq 4$ as $G_3 = 0$). Thus if $\tau \in \mathbb{H}$, we denote

$$G_k(\tau) = G_k(\Lambda(\tau)) = \sum_{m,n \in \mathbb{Z}^2 \setminus (0,0)} \frac{1}{(m+n\tau)^k}.$$

***Proposition 3.2.*** — *The function $G_k$ is a modular form of weight $k$ for $k \geq 4$ and even. Besides the value $G_k(\infty) = 2\zeta(k) = \sum_{m \in \mathbb{Z}^*} m^{-k}$*

*Proof.* — First let us check that $G_k$ is weakly modular. For this we need to study the transformation of $G_k$ under the action of $T$ and $S$.

$$\begin{aligned} G_k(\tau+1) &= \sum_{(m,n) \in \mathbb{Z}^2 \setminus (0,0)} \frac{1}{(m+n(\tau+1))^k} \\ &= \sum_{m \in \mathbb{Z}^*} \frac{1}{m^k} + \sum_{n \in \mathbb{Z}^*} \sum_{m \in \mathbb{Z}} \frac{1}{(m+n+n\tau)^k} \\ &= \sum_{m \in \mathbb{Z}^*} \frac{1}{m^k} + \sum_{n \in \mathbb{Z}^*} \sum_{m \in \mathbb{Z}} \frac{1}{(m+n\tau)^k} = G_k(\tau). \end{aligned}$$

The transformation under $S$ is similar

$$\begin{aligned} G_k(-1/\tau) &= \sum_{(m,n) \in \mathbb{Z}^2 \setminus (0,0)} \frac{1}{(m+n/\tau)^k} \\ &= \tau^k G_k(\tau). \end{aligned}$$

The convergence of $G_k$ at infinity is proven in Exercise **??**. In fact more is true, the series defining $G_k$ converges uniformly on the disk $\Delta_{\exp(-2\pi)}$ thus we can exchange sums and limit and we can see that as $\tau$ goes to infinity the summands tend to $1/m^k$ or 0. □

The standard renormalization of these modular forms is $E_k = \frac{G_k}{2\zeta(k)}$ (so that the value at infinity is 1). We will see in the next section that Eisentein series are not only good examples of modular forms but in some sense the only ones.

**3.3. Exercises.** —

***Exercise 3.3.*** — Let $A, B$ be positive real numbers, we denote by $\mathbb{H}_{A,B}$ the set of complex numbers such that $|z| > A$ and $|\Re(z)| < B$.

(1) Show that there exists $C > 0$ such that $|az+b| > C\max(|a|,|b|)$ for all $(a,b) \in \mathbb{R}^2 \setminus (0,0)$.

(2) Deduce that $\sum_{(n,m)} \frac{1}{|m\tau+n|^k} < \frac{1}{C^k} \sum_{s \geq 1} \frac{8s}{s^k}$. Use this to show that the Eisenstein series converge normally.

***Exercise 3.4.*** — (1) Show the following equality

$$\pi \cot(\pi z) = \frac{1}{z} + \sum_{n \geq 1} \frac{1}{z-n} + \frac{1}{z+n} = \pi i - 2\pi i \sum_{n \geq 0} \exp(2i\pi n z).$$

(2) Derive this formula to get the equality:

$$\sum_{n\in\mathbb{Z}} \frac{1}{(z+n)^k} = \frac{(-2i\pi)^k}{(k-1)!} \sum_{n\geq 1} q^n.$$

(3) Deduce from this identity that $G_k = 2\zeta(k) + \frac{2(2i\pi)^k}{(k-1)!} \sum_{n\geq 1} \sigma_{k-1}(n) q^n$, where $\sigma_{k-1}(n) = \sum_{m|n} m^{k-1}$.

## 4. The ring structure of $M_*$

The purpose of this section is to compute the dimension of the $M_k$ and describe the structure of $M_*$.

**4.1. Cuspidal forms.** — We denote by $\mathrm{ev} : M_k \to \mathbb{C}$ the evaluation of a modular form at $\infty$. This is a linear morphism and we denote by $S_k$ its kernel. The modular forms in $S_k$ are called *cuspidal.*

An important example of cuspidal form is the *Jacobi function* $\Delta = \frac{1}{1728}(E_4^3 - E_6^2)$. We will show further that $\Delta$ is not trivial. Indeed its value at infinity is 0, so we only need to show that another one is not 0. As $E_6(i) = 0$ we just need to show that $E_4(i) \neq 0$. Assuming this fact, we will have that $M_k \simeq S_{k+12}$ the isomorphism being given by $f \mapsto f \times \Delta$. It will be proved in the next section.

**4.2. Counting the zeros of $f$.** — Let $f$ be a modular form of weight $k$. If $\tau$ is a point in $\mathbb{H}$, we denote by $v_\tau(f)$ the order of $f$ at $\tau$ (the order of the first non-zero coefficient of $f$ if we develop it as a power series around $\tau$). Remark that the orders of $f$ at $\tau$ and $\gamma(\tau)$ are the same. Besides we denote by $v_\infty(f)$ the order of the $q$-expansion of $f$ at 0.

As $f$ is holomorphic on $\mathbb{H}$ and at $\infty$, it has finitely many zeros in $\overline{D}_{Dir}$. The following lemma is fundamental.

***Lemma 4.1.*** — *We have the following equality*

$$v_\infty(f) + \frac{v_i(f)}{2} + \frac{v_\rho(f)}{3} + \sum_\tau v_\tau(f) = k/12$$

*where the sum is over the classes of points in* $\overline{D}_{Dir} \setminus \{i, \rho\}$

*Proof.* — To prove this formula one needs to compute the integral of $f'/f$ along the contour ABCD of Figure 1 (During the lectures). □

Using this Lemma we can compute the dimensions $M_k$ for $k \leq 10$.

***Lemma 4.2.*** — *If* $k = 0, 4, 6, 8, 10$ *then* $M_k$ *is of dimension* 1, *for all other* $k \leq 10$ *we have* $M_k = \{0\}$.

*Proof.* — If $k \leq 2$, then the sum of the orders of zeros (with weights) if $\leq 1/6$. This implies that $f$ has no zeros. Thus by the maximum principle (see Lemma 2.2), $f$ is a constant.

Now let $f$ be modular form of weight 4, we can see that the sum of orders of zeros of $f$ is $\frac{1}{3}$. Therefore $v_\rho(f) = 1$ and $f$ has no zeros outside. Therefore $f - f(\infty)E_4$ is a modular form of weight 4 with a zero at $\infty$ and thus is 0. Therefore $M_4 = \mathbb{C} \cdot E_4$.

Now if $f$ is of weight 6, then $f$ has only one simple zero at $i$ and by the same line of arguments $M_6 = \mathbb{C} \cdot E_6$.

Following the same line of arguments we can check that a form of weight 8 has only one double zero at $\rho$ and a form of weight 10 has one simple zero at $i$ and one at $\rho$. Therefore $M_8 = \mathbb{C} \cdot E_4^2$ and $M_{10} = \mathbb{C} \cdot E_4 E_6$. □

The second corollary of this fundamental lemma is the structure of the ring of cuspidal modular forms.

***Lemma 4.3.*** — *We have the isomorphism $S_{k+12} \simeq M_k$*

*Proof.* — Indeed as explained in the previous section we just need to show that $E_4(i) \neq 0$ which is true as $E_4$ has only one zero at $\rho$. □

The conclusion of both lemmas is that the ring of modular form is in fact isomorphic to the ring $\mathbb{C}[E_4, E_6]$ (see Exercise 4.4).

### 4.3. Exercises. —

***Exercise 4.4.*** — We have shown above that $M_*$ is generated by $E_4$ and $E_6$ thus $M_*$ is isomorphic to $\mathbb{C}[E_4, E_6]/I$ where $I$ is an ideal. Show that $M_*$ is isomorphic to $\mathbb{C}[E_4, E_6]$ (or that $I$ is trivial, i.e. there is no algebraic relations between $E_4$ and $E_6$). *Hint: compute the dimensions of the $M_k$ and show that they agree with the dimensions of the subspaces of $\mathbb{C}[E_4, E_6]$ of polynomials of fixed degrees (with the degree of $E_k$ being $k$).*

## 5. Relation with elliptic curves

An *elliptic curve* is the zero locus in $\mathbb{C}^2$ of polynomial equation of degree 3. We will see in this section that a lattice $\Lambda$ of $\mathbb{C}$ gives a natural elliptic curve isomorphic (as a Riemann surface) to the quotient $\mathbb{C}/\Lambda$.

**5.1. Periodic functions.** — Let us fix a lattice $\Lambda = \omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z}$ of $\mathbb{C}$. The problem of this section is the following: *does there exist periodic meromorphic functions defined on $\mathbb{C}$ with periods $\omega_1$ and $\omega_2$?* (i.e. $f(z + \omega_1) = f(z + \omega_2) = f(z)$).

In order to answer this question we follow the most natural approach, i.e. we average some meromorphic functions with the lattice $\Lambda$:

$$\begin{aligned} \wp(z) &= \frac{1}{z^2} + \sum_{\omega \in \Lambda^*} \frac{1}{(z+\omega)^2} - \frac{1}{\omega^2} \\ \widetilde{\wp}(z) &= \sum_{\omega \in \Lambda} \frac{1}{(z+\omega)^3} \end{aligned}$$

(we could not simply write $\wp(z) = \sum_{\omega \in \Lambda} \frac{1}{(z+\omega)^2}$ because we wanted to have the normal convergence of the series). The function $\wp$ is called the *Weierstrass function of* $\Lambda$. We can easily check that $\widetilde{\wp} = -2\wp'$.

Now, we denote by $\mathcal{M}(\Lambda)$ the field of periodic meromorphic functions for the lattice $\Lambda$ (check that the products, inverse and sums of periodic meromorphic functions is still periodic). We will show

***Proposition 5.1.*** — *The field $\mathcal{M}(\Lambda)$ is isomorphic to*

$$\mathbb{C}(\wp, \wp')/(\wp'^2 - 4\wp^3 - g_2(\Lambda)\wp - g_3(\Lambda)$$

*where $g_2 = 60G_4$ and $g_3 = 140G_6$. In particular $\mathbb{C}/\Lambda$ is isomorphic to the (smooth) elliptic curve defined by the equation $Y^2 = 4X^3 + g_2X + g_3$*

*Proof.* — Begin by integrating $f$ and $f'/f$ along the contour of a fundamental domain of the lattice (during the lectures). □

## 6. Hecke Operators

In this last section, the purpose is to prove an observation made by Ramanujan regarding the $q$-expansion of the Jacobi function (see [?]). If we write $\Delta = \sum_{n\geq 0} \tau(n)q^n$, then

- $\tau(mn) = \tau(m)\tau(n)$ if $m$ and $n$ are co-prime;
- $\tau(p^{r+1}) = \tau(p)\tau(p^{r+1}) + p^{11}\tau(p^{r-1})$ if $p$ is prime.

**6.1. Construction of Hecke operators.** — We denote by $\mathcal{R} = \bigoplus_\Lambda \mathbb{C}[\Lambda]$ the vector space with basis indexed by lattices of $\mathbb{C}$. For all $n \geq 1$, we define the following operators of $\mathcal{R}$:

$$\begin{aligned} T_n([\Lambda]) &= \sum_{[\Lambda':\Lambda]=n} [\Lambda'] \\ R_n([\Lambda]) &= [n\Lambda] \end{aligned}$$

where the first sum is over sub-lattices of index $n$. This sum is finite as the group $\Lambda/n\lambda$ is isomorphic to $(\mathbb{Z}/n\mathbb{Z})^2$ and finite. These operators have the following properties:

- $R_n \circ R_m = R_{nm}$;
- $R_n \circ T_m = T_m \circ R_n$;
- $T_{nm} = T_n \circ T_m$ if $(n, m) = 1$;
- $T_{p^{r+1}} = T_{p^r} \circ T_p - p \cdot R_p \circ T_{p^{r-1}}$ if p is prime.

(see Exercise 6.3). These operators acts on the set of functions on lattices by

$$(T_n(F))(\Lambda) = \sum_{[\Lambda':\Lambda]=n} (T_n(F))(\Lambda')$$

(and the same for $R_n$). Now, if $f$ is a modular form of weight $k$, then it defines naturally a function $F$ on the space of lattices by setting $F(\omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z}) = \omega_2^{-k} f\left(\frac{\omega_1}{\omega_2}\right)$ (if we assume that $\omega_1/\omega_2 \in \mathbb{H}$). This function $F$ is *homogeneous of weight* $k$, i.e. $F(\lambda\Lambda) = \lambda^{-k}F(\Lambda)$. Conversely an homogeneous functions determines uniquely a modular from. Thus we can define the action of the Hecke operators on weakly modular functions f weight $k$ by

$$T_n(f)(\tau) = n^{k-1}F(\tau\mathbb{Z} \oplus \mathbb{Z})$$

(the factor $n^{k-1}$ is arbitrary but standard in the literature).

**6.2. Action of the Hecke operators on the $q$-expansion.** —

***Proposition 6.1.*** — *The spaces $M_k$ and $S_k$ are invariant under the action of the operators $T_n$ and $R_n$.*

*Proof.* — It is obvious from the definition of $T_n$ and $R_n$ that these operators preserve the homogeneity of a function defined on lattices, thus it preserves the weakly modular property.

The operator $R_n$ acts by multiplication of modular form of weight $k$ by $n^{-2k}$. To show that the analicity of a function at $\infty$ is preserved, we will describe more precisely the action of $T_n$ on modular forms. For this we will use the following lemma.

***Lemma 6.2.*** — *Any lattice of index $n$ in $\Lambda(\tau)$ is of the form $(a\tau+b)\mathbb{Z}\oplus d\mathbb{Z}$ where $ad=n$, $a>0$, and $0\leq b<d$.*

*Proof.* — It is obvious that all $(a\tau+b)\mathbb{Z}\oplus d\mathbb{Z}$ where $ad=n$, $a>0$, and $0\leq b<d$ are sublattices of index $n$. Besides we can obviously check that two such lattices are distinct. To show that any sub-lattice is of this form, let $\Lambda'$ be a sub-lattice of $\Lambda$ of index $n$, we define $d$ to be the minimal integer such that $d\in\Lambda'$. Then $\Lambda'=\omega\mathbb{Z}\oplus d\mathbb{Z}$ with $\omega=az+b$. We can check that $ad=n$ and that $b$ can be chosen in $[\![0,d-1]\!]$. □

In particular, we can rewrite

$$T_n(f)(\tau)=n^{k-1}\sum_{d|n}\sum_{0\leq b<d}d^{-k}f(\tau+\frac{b}{d})$$

In particular, if $f$ is a modular expansion, then $T_n(f)$ is analytic (thus a quasi-modular form) and its $q$-expansion is given by

$$T_n(f)(\tau)=\sum_{m\geq 0}\left(\sum_{d|(m,n)}d^{k-1}a_{mn/d^2}\right)q^m.$$

Thus $T_n(f)\in S_k$ iff $f\in S_k$. □

We obtain the identity for the tau-function by remarking that $S_{12}$ is 1-dimensional, thus $\Delta$ is an eigenvector of all operators $T_n$. Besides the explicit expression of $T_n(\Delta)$ shows that the coefficient in front of $q$ is equal to $\tau_n$ thus the eigenvalue of $\Delta$ for $T_n$ is $\tau_n$. The identities for the tau function then follow from the identities for the operators $T_n$ and $R_n$.

### 6.3. Exercises. —

***Exercise 6.3.*** — Let $\Gamma$ be a lattice and let $\Gamma'$ be a sub-lattice of index $p^{n+1}$. We denote by $a$ the coefficient of $\Gamma'$ in $T_{p^n}\circ T_p(\Gamma)$ and by $c$ the coefficient of $\Gamma'$ in $T_{p^{n-1}}\circ R_p(\Gamma)$.

(1) If $\Gamma'$ is not included in $p\Gamma$ then show that $a=1$ and $c=0$.

(2) If $\Gamma'$ is included in $p\Gamma$, show that $a=p+1$ and $c=1$. Conclude that the identity $T_{p^{r+1}}=T_{p^r}\circ T_p-p\cdot R_p\circ T_{p^{r-1}}$ holds for all primes $p$.

## References

[BvdGHZ08] Jan Hendrik Bruinier, Gerard van der Geer, Günter Harder, and Don Zagier. *The 1-2-3 of modular forms*. Universitext. Springer-Verlag, Berlin, 2008. Lectures from the Summer School on Modular Forms and their Applications held in Nordfjordeid, June 2004, Edited by Kristian Ranestad.

[J.S15] Milne J.S. Modular functions and modular forms. 2015, www.jmilne.org/math/CourseNotes/MF.pdf.

[Ser73] J.-P. Serre. *A course in arithmetic*. Springer-Verlag, New York-Heidelberg, 1973. Translated from the French, Graduate Texts in Mathematics, No. 7.

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