Asymptotics of quantum invariants

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Part I

Perspective

Perspective: the Kashaev volume conjecture

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Murakami-Murakami 1999: If $J_L^n(q) \in \mathbb{Z}[q, q^{-1}]$ is the (normalized) n–th colored Jones polynomial of $L \subset S^3$,

$$K_n(S^3, L) = J_L^n\left(\mathrm{e}^{\frac{2\pi \mathrm{i}}{n}}\right)$$

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- still pretty amazing. Example: A quantum invariant which is a sum of 10^{139} terms (corresponding to states), many of which are of the order of 10^{35}; however, the sum is only of the order of 10^{22}. The cancellations are not algebraic term-by-term, but "on average".

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Part II

The SL_2–skein algebra and its representations

The SL_2–skein algebra of a surface

For $q \in \mathbb{C}^*$, the SL_2*–skein algebra* of a surface S is

$$\mathcal{S}^q_{\mathrm{SL}_2}(S) = \left\{\mathbb{C}\text{–linear comb. of framed links } L \subset S \times [0,1]\right\}/\text{skein relations}$$

SL_2–skein relations:

-  $= q^{\frac{1}{2}}$  $+ q^{-\frac{1}{2}}$ 
-  $= -\left(q + q^{-1}\right) \varnothing$

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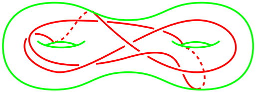

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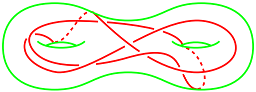

Multiplication by superposition: If $[L_1]$, $[L_2] \in \mathcal{S}^q_{\mathrm{SL}_2}(S)$,
$[L_1] \cdot [L_2] = [L_1 \sqcup L_2]$ where $L_1 \subset S \times [0, \frac{1}{2}]$ and $L_2 \subset S \times [\frac{1}{2}, 1]$

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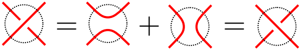
$q = 1 \Longrightarrow$ 

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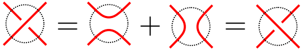
$q = 1 \Longrightarrow$ 

$\Longrightarrow \mathcal{S}^q_{\mathrm{SL}_2}(S)$ can be seen as a deformation of the algebra $\mathcal{S}^1_{\mathrm{SL}_2}(S)$ of regular functions on the $\mathrm{SL}_2(\mathbb{C})$–character variety

$$\mathcal{X}_{\mathrm{SL}_2}(S) = \Big\{\text{group hom. } r\colon \pi_1(S) \to \mathrm{SL}_2(\mathbb{C})\Big\} /\!\!/ \mathrm{SL}_2(\mathbb{C})$$

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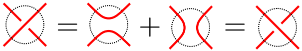
$q = 1 \Longrightarrow$ 

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Proposition (Helling?)

For every algebra homomorphism $\rho\colon \mathcal{S}^1_{\mathrm{SL}_2}(S) \to \mathbb{C}$, there exists a unique $r \in \mathcal{X}_{\mathrm{SL}_2}(S)$ such that

$$\rho([K]) = -\mathrm{Trace}\, r(K)$$

for every knot $K \subset S \times [0,1]$

The G–skein algebra of a surface

More generally, if G is a Lie group, with associated quantum group $\mathrm{U}_q(\mathfrak{g})$, the *$G$–skein algebra* $\mathcal{S}^1_G(S)$ is

$$\mathcal{S}^q_G(M) = \left\{\begin{array}{l}\text{free } \mathbb{C}\text{–vector space} \\ \text{generated by links } L \subset M \\ \text{colored by reps of } \mathrm{U}_q(\mathfrak{g})\end{array}\right\} \Big/ \left\{\begin{array}{l}\text{all relations between} \\ \text{tensor products of} \\ \text{representations of } \mathrm{U}_q(\mathfrak{g})\end{array}\right\}$$

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When $G = \mathrm{SL}_d = \{d \times d \text{ matrices with } \det = 1\}$ and q generic, Cautis-Kamnitzer-Morrison give explicit relations in terms of colored trivalent graphs called webs

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Remark: For the geometrically inclined, it may be better to replace representations of the *quantum group* $\mathrm{U}_q(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ by co-representations of the *quantum coordinate ring* $\mathcal{O}_q(G)$ of the Lie group G. Geometers tend to have more intuition for Lie groups than for Lie algebras

The center of SL_2–skein algebra for q generic

S oriented surface of finite topological type, possibly with punctures

Theorem (Etingof?)

When q is not a root of unity, the center of $\mathcal{S}^q_{\mathrm{SL}_2}(S)$ is generated by simple loops $P_v \subset S \times \frac{1}{2}$ going around the punctures v of S

The center of SL_2–skein algebra for $q^n = 1$

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$\mathrm{Trace}\, A^n = T_n(\mathrm{Trace}\, A)$ for every $A \in \mathrm{SL}_2$

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Theorem (FB + Helen Wong)

When $q^n = 1$ primitive with n odd and $(q^{\frac{1}{2}})^n = -1$, there is a unique central Frobenius embedding
$$\mathbf{F} \colon \mathcal{S}^1_{\mathrm{SL}_2}(S) \to \mathcal{S}^q_{\mathrm{SL}_2}(S)$$
such that
$$\mathbf{F}\big([K]\big) = T_n\big([K]\big)$$
for every simple closed curve $K \subset S \times \frac{1}{2} \subset S \times [0,1]$

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Frohman-Kania-Bartoszyńska-Lê: The center of $\mathcal{S}^q_{\mathrm{SL}_2}(S)$ is generated by the image $\mathbf{F}\big(\mathcal{S}^1_{\mathrm{SL}_2}(S)\big)$ and the loops P_v around the punctures

Invariants of representations of $\mathcal{S}^q_{\mathrm{SL}_2}(S)$

Corollary

If $(q^{\frac{1}{2}})^n = -1$ with n odd and $\rho\colon \mathcal{S}^q_{\mathrm{SL}_2}(S) \to \mathrm{End}(V)$ is an irreducible representation of $\mathcal{S}^q_{\mathrm{SL}_2}(S)$, there exists a unique $r_\rho \in \mathcal{X}_{\mathrm{SL}_2}(S)$ and weights $p_v \in \mathbb{C}$ associated to the punctures v of S such that

- $\rho([K]) = -\mathrm{Trace}\, r_\rho([K])\,\mathrm{Id}_V$ *for every simple closed curve* $K \subset S \times \frac{1}{2} \subset S \times [0,1]$
- $\rho([P_v]) = p_v\,\mathrm{Id}_V$ *for every simple loop* P_v *going around the puncture* v

In addition, the puncture weights $p_v \in \mathbb{C}$ are compatible with r in the sense that $T_n(p_v) = -\mathrm{Trace}\, r_\rho(P_v)$ for every puncture v

$r_\rho \in \mathcal{X}_{\mathrm{SL}_2}(S)$ is the *classical shadow* of the representation $\rho\colon \mathcal{S}^q_{\mathrm{SL}_2}(S) \to \mathrm{End}(V)$

Classification of “most” representations of $\mathcal{S}^q_{\mathrm{SL}_2}(S)$

Theorem (Frohman-Kania-Bartoszyńska-Lê, Ganev-Jordan-Safronov, Detcherry-Santharoubane)

Suppose $(q^{\frac{1}{2}})^n = -1$ with n odd. For every smooth point $r \in \mathcal{X}_{\mathrm{SL}_2}(S)$ and every system of puncture weights $p_v \in \mathbb{C}$ compatible with r there exists, up to isomorphism, a unique irreducible representation $\rho\colon \mathcal{S}^q_{\mathrm{SL}_2}(S) \to \mathrm{End}(V)$ whose classical shadow and puncture invariants are r and the p_v
In addition, $\dim V = n^{3g-3+p}$

Remark. By Ganev-Jordan-Safronov and more recent results (FB + Higgins, Higgins, Z. Wang, H. K. Kim, Lê, Karuo, . . .), similar properties hold, or should hold, for $\mathcal{S}^q_{\mathrm{SL}_d}(S)$

Part III

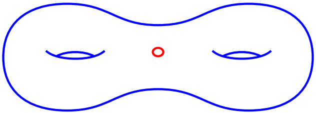
One more volume conjecture

Volume conjecture for surface diffeomorphisms

Based on joint work with Helen Wong and Tian Yang

Invariant $\mathrm{SL}_2(\mathbb{C})$–characters

S oriented surface of finite topological type, with genus g and $p \geqslant 0$ punctures



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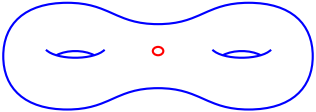

A diffeomorphism $\varphi\colon S \to S$ acts on the character variety $\mathcal{X}_{\mathrm{SL}_2}(S)$, on the skein algebra $\mathcal{S}^{q}_{\mathrm{SL}_2}(S)$, and on representations of the skein algebra

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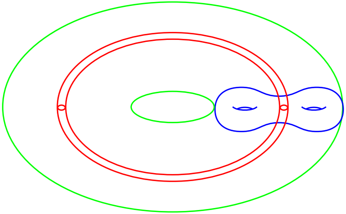
$$M_\varphi = S \times [0,1]/(x,1) \sim (\varphi(x),0)$$



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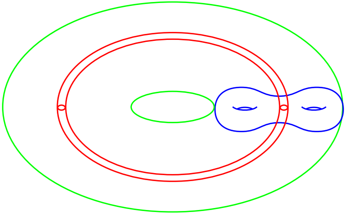

- If S has p punctures, the fixed point set of the action of φ is a smooth submanifold of complex dimension p near $r^{\mathrm{hyp}} \in \mathcal{X}_{\mathrm{SL}_2}(S)$

Invariant representations of $\mathcal{S}^q_{\mathrm{SL}_2}(S)$

Recall. Suppose $(q^{\frac{1}{2}})^n = -1$ with n odd. For every smooth point $r \in \mathcal{X}_{\mathrm{SL}_2}(S)$ and every system of puncture weights $p_v \in \mathbb{C}$ such that $T_n(p_v) = -\mathrm{Trace}\, r\big([P_v]\big)$ for every puncture v, there exists, up to isomorphism, a unique irreducible representation $\rho\colon \mathcal{S}^q_{\mathrm{SL}_2}(S) \to \mathrm{End}(V)$ whose classical shadow and puncture weights are r and the p_v

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- system of puncture weights $p_v \in \mathbb{C}$ that is *compatible with* r in the sense that $T_n(p_v) = -\mathrm{Trace}\, r(P_v)$ for every v, and *φ–invariant* in the sense that $p_{\varphi(v)} = p_v$ for every v

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Then, the theorem associates to this φ–invariant data a representation $\rho\colon \mathcal{S}^q_{\mathrm{SL}_2}(S) \to \mathrm{End}(V)$

Invariant representations of $\mathcal{S}^q_{\mathrm{SL}_2}(S)$

Recall. Suppose $(q^{\frac{1}{2}})^n = -1$ with n odd. For every smooth point $r \in \mathcal{X}_{\mathrm{SL}_2}(S)$ and every system of puncture weights $p_v \in \mathbb{C}$ such that $T_n(p_v) = -\mathrm{Trace}\, r\big([P_v]\big)$ for every puncture v, there exists, up to isomorphism, a unique irreducible representation $\rho\colon \mathcal{S}^q_{\mathrm{SL}_2}(S) \to \mathrm{End}(V)$ whose classical shadow and puncture weights are r and the p_v

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Invariant representations of $\mathcal{S}^q_{\mathrm{SL}_2}(S)$

The φ–invariance of the representation $\rho\colon \mathcal{S}^q_{\mathrm{SL}_2}(S) \to \mathrm{End}(V)$ means that there exists a linear isomorphism $\Lambda^q_{\varphi,r,p_v}\colon V \to V$ such that

$$\rho\big(\varphi[L]\big) = \Lambda^q_{\varphi,r,p_v} \circ \rho\big([L]\big) \circ \big(\Lambda^q_{\varphi,r,p_v}\big)^{-1} \text{ in } \mathrm{End}(V)$$

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Proposition

$\left|\mathrm{Trace}\,\Lambda^q_{\varphi,r,p_v}\right|$ *depends only on q with $(q^{\frac{1}{2}})^n = -1$ and n odd, on the φ–invariant character $r \in \mathcal{X}_{\mathrm{SL}_2}(S)$ and on the φ–invariant compatible puncture weights $p_v \in \mathbb{C}$*

The Chebyshev equation

Compatibility equation. The puncture weights p_v must satisfy

$$T_n(p_v) = -\mathrm{Trace}\, r\big([P_v]\big)$$

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Lemma

If λ_v and λ_v^{-1} are the eigenvalues of $r\big([P_v]\big) \in \mathrm{SL}_2(\mathbb{C})$, the solutions of the equation

$$T_n(p_v) = -\mathrm{Trace}\, r\big([P_v]\big) \ (= \lambda_v + \lambda_v^{-1})$$

are the numbers of the form

$$p_v = -\lambda_v^{\frac{1}{n}} - \lambda_v^{-\frac{1}{n}}$$

as $\lambda_v^{\frac{1}{n}}$ franges over all n–roots of λ_v

The volume conjecture for surface diffeomorphisms

Geometric data.

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- a diffeomorphism $\varphi\colon S \to S$

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Conjecture

$$\lim_{n \text{ odd} \to \infty} \frac{1}{n} \log \left| \mathrm{Trace}\, \Lambda^q_{\varphi, r, p_v} \right| = \frac{1}{4\pi} \mathrm{vol}_{\mathrm{hyp}}\, M_\varphi$$

where $\mathrm{vol}_{\mathrm{hyp}}\, M_\varphi$ is the volume of the hyperbolic metric of the mapping torus M_φ

The volume conjectures for surface diffeomorphisms

Suppose instead that

$$p_v = -q^{m_{v,n}} \mathrm{e}^{\frac{1}{n}\theta_v} - q^{-m_{v,n}} \mathrm{e}^{-\frac{1}{n}\theta_v}$$

for correction factors $m_{v,n} \in \mathbb{Z}$ with $m_{\varphi(v),n} = m_{v,n}$ and

$$\lim_{n \text{ odd} \to \infty} \frac{4\pi m_{v,n}}{n} = \alpha_v \in [0, 2\pi]$$

Conjecture (T. Pandey, K. H. Wong)

$$\lim_{n \text{ odd} \to \infty} \frac{1}{n} \log \left| \mathrm{Trace}\, \Lambda^q_{\varphi,r,p_v} \right| = \frac{1}{4\pi} \mathrm{vol}_{\mathrm{hyp}}\, M_\varphi(\alpha_v)$$

where $\mathrm{vol}_{\mathrm{hyp}}\, M_\varphi(\alpha_v)$ is the volume of the hyperbolic cone manifold obtained from M_φ by Dehn filling with cone angle $\alpha_v \in [0, 2\pi]$ along each cusp of M_φ corresponding to the φ–orbit of the puncture v.

Numerical evidence

$S =$ one-puncture torus

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Pick some φ–invariant character $r \in \mathcal{X}_{\mathrm{SL}_2}(S)$ and logarithm θ_v for the eigenvalues $\mathrm{e}^{\pm\theta_v}$ of $r(P_v) \in \mathrm{SL}_2(\mathbb{C})$

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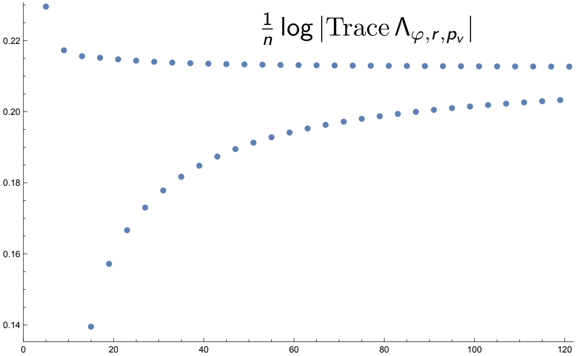
For every n odd and puncture weight $p_v = -\mathrm{e}^{\frac{1}{n}\theta_v} - \mathrm{e}^{-\frac{1}{n}\theta_v}$, the machinery gives us an isomorphism $\Lambda_{\varphi,r,p_v} \colon V \to V$ with $\dim V = n$

Numerical evidence



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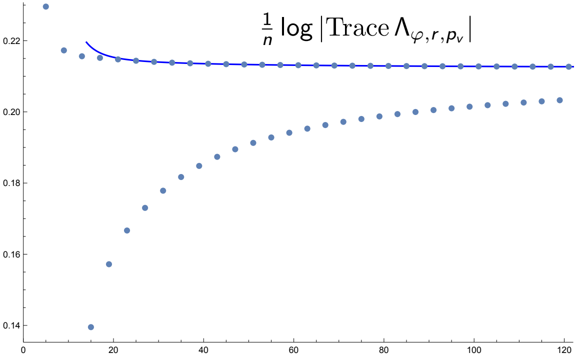

Top $\approx 0.212253 - \frac{1.07278}{n} + \frac{0.715999}{n^2} - \frac{40.8161}{n^3} + \frac{656.735}{n^4}$

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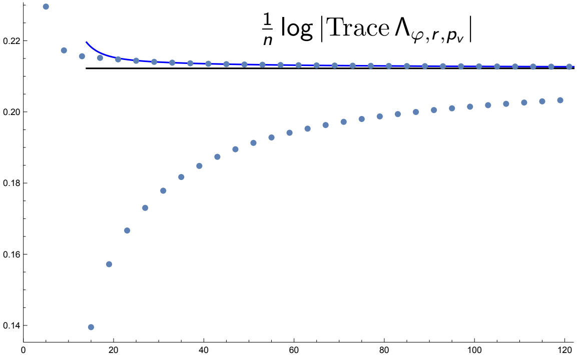

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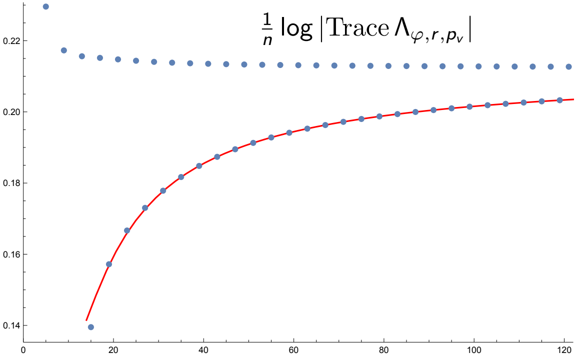

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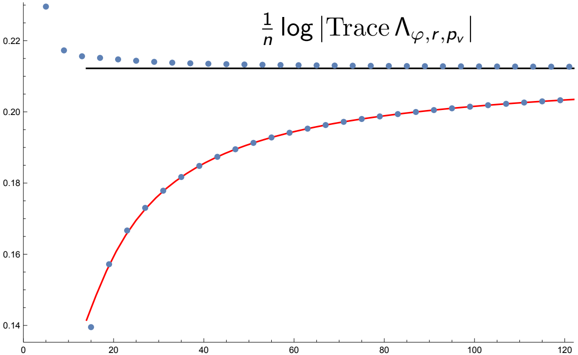

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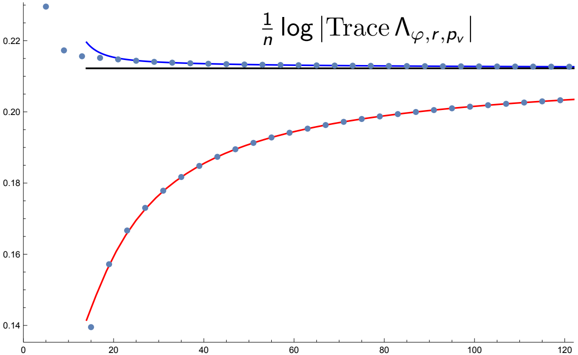

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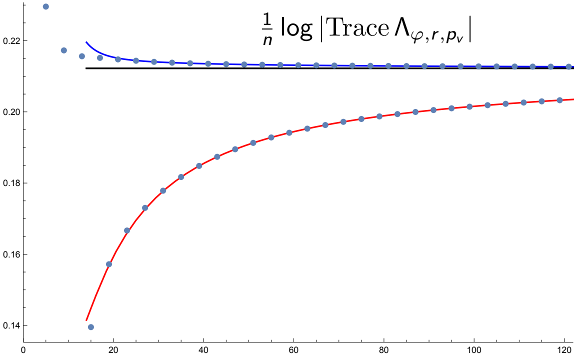

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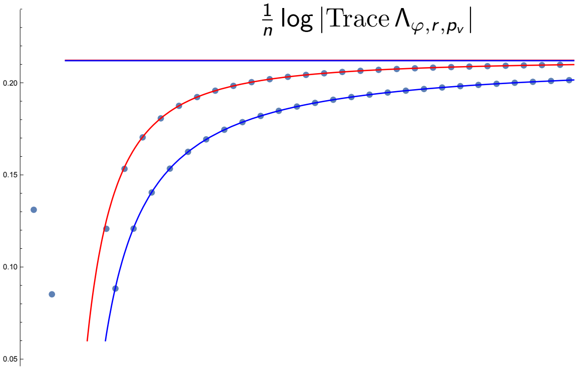
$$\frac{1}{4\pi}\mathrm{vol}_{\mathrm{hyp}}(M_\varphi) \approx 0.212212$$

Numerical evidence

An example with the same $\varphi\colon S \to S$, but a different φ–invariant character $r \in \mathcal{X}_{\mathrm{SL}_2}(S)$



Part IV

How to compute

The quantum Teichmüller space

Recall. The SL_2–skein algebra $\mathcal{S}^q_{\mathrm{SL}_2}(S)$ is a deformation of the algebra of functions on the $\mathrm{SL}_2(\mathbb{C})$–character variety

$$\mathcal{X}_{\mathrm{SL}_2}(S) = \left\{\text{group hom. } r\colon \pi_1(S) \to \mathrm{SL}_2(\mathbb{C})\right\} /\!\!/ \mathrm{SL}_2(\mathbb{C})$$

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The *quantum Teichmüller space* is a deformation of the algebra of functions on the *enhanced* $\mathrm{PSL}_2(\mathbb{C})$*–character variety*

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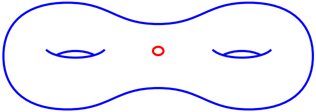
Advantages / drawbacks

- The skein algebra is very intrinsic, but hard to work with
- The quantum Teichmüller space is a conceptual mess, but easier to compute with

The quantum Teichmüller space

S = oriented surface of genus g with $p \geqslant 1$ punctures



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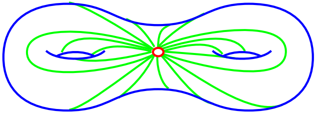

An *ideal triangulation* of S is a triangulation τ with all vertices at the punctures, with edges $e_1, e_2, \ldots, e_{6g-6+3p}$

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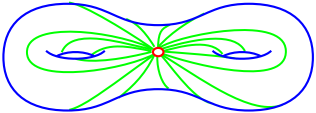

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Thurston used ideal triangulations to construct *shearbend coordinates* for the enhanced character variety $\mathcal{X}^{\mathrm{enh}}_{\mathrm{PSL}_2(\mathbb{C})}(S)$.

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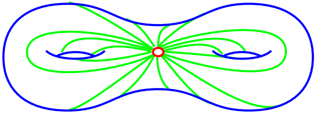

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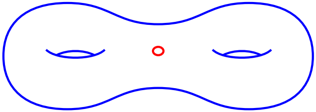
There is one coordinate $x_i \in \mathbb{C}^*$ for each edge of the ideal triangulation τ.
Each x_i is a crossratio of eigenlines associated to the punctures of S

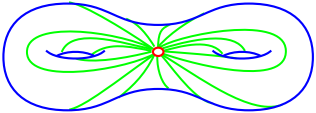
The quantum Teichmüller space



The *Chekhov-Fock algebra* of the ideal triangulation τ is the Laurent polynomial algebra

$$\mathcal{CF}^q_\tau = \mathbb{C}[X_1^{\pm 1}, X_2^{\pm 1}, \ldots, X_{6g-6+3p}^{\pm 1}]^q$$

The quantum Teichmüller space



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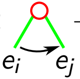
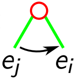
where $X_i X_j = q^{2\varepsilon_{ij}} X_j X_i$ with

$\varepsilon_{ij} = \sharp$ 

$- \sharp$ 

The quantum Teichmüller space

$\widehat{\mathcal{CF}^q_\tau}$ = fraction algebra of $\mathcal{CF}^q_\tau$

Theorem (Chekhov-Fock + H. Bai)

Up to uniform rescaling of the X_i, there exists a unique family of algebra isomorphisms

$$\Psi^q_{\tau\tau'} : \widehat{\mathcal{CF}^q_{\tau'}} \to \widehat{\mathcal{CF}^q_\tau}$$

as τ, τ' range over all ideal triangulations of S

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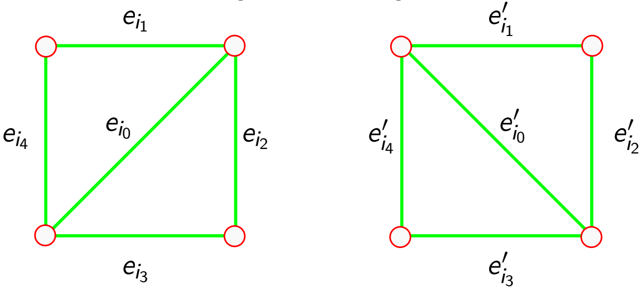
as τ, τ' range over all ideal triangulations of S, such that

$$\Psi^q_{\tau\tau''} = \Psi^q_{\tau\tau'} \circ \Psi^q_{\tau'\tau''}$$

for any three ideal triangulations τ, τ', τ''

The quantum Teichmüller space

Fundamental case: the diagonal exchange



$$\Psi^q_{\tau\tau'}(X'_i) = \begin{cases} X_{i_0}^{-1} & \text{if } i = i_0 \\ X_i\left(1 + q^{-1}X_{i_0}\right) & \text{if } i = i_1 \text{ or } i_3 \\ X_i\left(1 + q^{-1}X_{i_0}^{-1}\right)^{-1} & \text{if } i = i_2 \text{ or } i_4 \\ X_i & \text{if } i \neq i_0, i_1, i_2, i_3, i_4 \end{cases}$$

The quantum Teichmüller space

The *quantum Teichmüller space* $\mathcal{T}^q(S)$ of S is the family of the Chekhov-Fock algebras $\mathcal{CF}^q_\tau$ and of the quantum coordinate changes $\Psi^q_{\tau\tau'}$

The quantum Teichmüller space

The *quantum Teichmüller space* $\mathcal{T}^q(S)$ of S is the family of the Chekhov-Fock algebras $\mathcal{CF}^q_\tau$ and of the quantum coordinate changes $\Psi^q_{\tau\tau'}$

When $q = 1$, this corresponds to Thurston's shearbend coordinates for the *enhanced character variety* $\mathcal{X}^{\mathrm{enh}}_{\mathrm{PSL}_2(\mathbb{C})}(S)$, consisting of homomorphisms $r\colon \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$ enhanced with the data of an eigenline for $r(P_v) \in \mathrm{PSL}_2(\mathbb{C})$ at each puncture v

Representations of the quantum Teichmüller space

A *representation* $\bar{\rho}\colon \mathcal{T}^q(S) \to \mathrm{End}(V)$ of the quantum Teichmüller space is a family of representations $\bar{\rho}_\tau\colon \mathcal{CF}^q_\tau \to \mathrm{End}(V)$, as τ ranges over all ideal triangulations,

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$$\bar\rho_{\tau'}(X') = \bar\rho_\tau\big(\Psi^q_{\tau\tau'}(X')\big) = \bar\rho_\tau(P)\circ\bar\rho_\tau(Q)^{-1} = \bar\rho_\tau(Q')^{-1}\circ\bar\rho_\tau(P')$$

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Representations of the quantum Teichmüller space

Recall:

Theorem (FB + Helen Wong 2016)

When $q^n = 1$ primitive with n odd and $(q^{\frac{1}{2}})^n = -1$, there is a unique central Frobenius embedding

$$\mathbf{F}\colon \mathcal{S}^1_{\mathrm{SL}_2}(S) \to \mathcal{S}^q_{\mathrm{SL}_2}(S)$$

such that

$$\mathbf{F}\big([K]\big) = T_n\big([K]\big)$$

for every simple closed curve $K \subset S \times \frac{1}{2} \subset S \times [0,1]$

Frohman-Kania-Bartoszyńska-Lê: The center of $\mathcal{S}^q_{\mathrm{SL}_2}(S)$ is generated by the image $\mathbf{F}\big(\mathcal{S}^1_{\mathrm{SL}_2}(S)\big)$ and the loops P_v around the punctures

Representations of the quantum Teichmüller space

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$$\mathbf{F}_\tau \colon \mathrm{CF}^1_\tau \to \mathcal{CF}^q_\tau$$

defined $\mathbf{F}_\tau(X_i) = X_i^n$ for every generator X_i. It is compatible with the quantum coordinate changes $\Psi^q_{\tau\tau'}$, in the sense that

$$\Psi^1_{\tau\tau'} \circ \mathbf{F}_{\tau'} = \mathbf{F}_\tau \circ \Psi^q_{\tau\tau'}$$

for every ideal triangulations τ, τ' of S

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Complement. The center of $\mathcal{CF}^q_\tau$ is generated by the image $\mathbf{F}\big(\mathrm{CF}^1_\tau\big)$ and by elements

$$H_v = q^{-\sum_{k<l} \varepsilon_{i_k i_l}} X_{i_1} X_{i_2} \dots X_{i_m}$$

associated to the punctures v, with e_{i_1}, e_{i_2}, ..., e_{i_m} the edges ending at v

Representations of the quantum Teichmüller space

Recall.

Theorem

Suppose that $q^n = 1$ *primitive with* n *odd and* $(q^{\frac{1}{2}})^n = -1$*. If* $\rho\colon \mathcal{S}^q_{\mathrm{SL}_2}(S) \to \mathrm{End}(V)$ *is an irreducible representation of* $\mathcal{S}^q_{\mathrm{SL}_2}(S)$*, there exists a unique* $r_\rho \in \mathcal{X}_{\mathrm{SL}_2}(S)$ *and puncture weights* $p_v \in \mathbb{C}$ *compatible with* r *such that*

- $\rho([K]) = -\mathrm{Trace}\, r_\rho([K])\, \mathrm{Id}_V$ *for every simple closed curve* $K \subset S \times \frac{1}{2} \subset S \times [0,1]$
- $\rho([P_v]) = p_v\, \mathrm{Id}_V$ *for every simple loop* P_v *going around the puncture* v

In addition, $\rho\colon \mathcal{S}^q_{\mathrm{SL}_2}(S) \to \mathrm{End}(V)$ *is uniquely determined by this data if* r *is a smooth point of the character variety* $\mathcal{X}_{\mathrm{SL}_2}(S)$

Representations of the quantum Teichmüller space

Theorem (FB + Xiaobo Liu 2007)

When $q^n = 1$ primitive with n odd, every irreducible representation

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- *an enhanced character $\bar r \in \mathcal{X}^{\mathrm{enh}}_{\mathrm{PSL}_2(\mathbb{C})}(S)$ such that, for every edge e_i of the ideal triangulation τ, $\bar\rho_\tau(X_i^n) = x_i \,\mathrm{Id}_V$ for the shearbend coordinate x_i of r along the edge e_i*

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- *a puncture invariant $h_v \in \mathbb{C}^*$ associated to each puncture v, such that $\bar\rho_\tau(H_v) = h_v \,\mathrm{Id}_V$ and $h_v^n = \lambda_v^2$ for the eigenvalue λ_v of $\bar r(P_v) \in \mathrm{PSL}_2(\mathbb{C})$ corresponding to the preferred eigenline given by the enhancement*

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Conversely, two representations of the quantum Teichmüller space are isomorphic if and only if they have the same classical shadow $\bar r \in \mathcal{X}^{\mathrm{enh}}_{\mathrm{PSL}_2(\mathbb{C})}(S)$ and puncture invariants $h_v \in \mathbb{C}^*$, and every data as above is realized by a representation

Quantum Teichmüller invariants of surface diffeomorphisms

A diffeomorphism $\varphi\colon S \to S$ induces a preferred isomorphism

$$\Phi^q_{\varphi(\tau)\tau} \colon \mathcal{CF}^q_\tau \to \mathcal{CF}^q_{\varphi(\tau)}$$

sending the generators X_i of $\mathcal{CF}^q_\tau$ associated to the edge e_i of τ to the generator X_i' of $\mathcal{CF}^q_{\varphi(\tau)}$ associated to the edge $e_i' = \varphi(e_i)$ of $\varphi(\tau)$

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This defines an action of φ on the quantum Teichmüller space

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Then, the theorem associates to this φ–invariant data an irreducible representation

$$\bar{\rho} = \{\bar{\rho}_\tau \colon \mathcal{CF}^q_\tau \to \mathrm{End}(V); \tau \text{ ideal triangulation}\}$$

of the quantum Teichmüller space which is φ–invariant up to isomorphism

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$$\bar{\rho}_{\varphi(\tau)} \circ \Phi^q_{\varphi(\tau)\tau}(X) = \bar{\Lambda}^q_{\varphi,\bar{r},h_v} \circ \bar{\rho}_\tau(X) \circ \bar{\Lambda}^q_{\varphi,\bar{r},h_v}{}^{-1} \in \mathrm{End}(V)$$

for every $X \in \mathcal{CF}^q_\tau(S)$, where $\Phi^q_{\varphi(\tau)\tau} \colon \mathcal{CF}^q_\tau \to \mathcal{CF}^q_{\varphi(\tau)}$ induced by φ

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for every $X \in \mathcal{CF}^q_\tau(S)$, where $\Phi^q_{\varphi(\tau)\tau} : \mathcal{CF}^q_\tau \to \mathcal{CF}^q_{\varphi(\tau)}$ induced by φ
Normalize so that $\det \bar{\Lambda}^q_{\varphi,\bar{r},h_v} = 1$

Proposition

$\left|\mathrm{Trace}\,\bar{\Lambda}^q_{\varphi,\bar{r}}\right|$ depends only on q with $q^n = 1$ and n odd, on the enhanced φ–invariant character $\bar{r} \in \mathcal{X}^{\mathrm{enh}}_{\mathrm{PSL}_2(\mathbb{C})}(S)$ and on the φ–invariant compatible puncture weights $h_v \in \mathbb{C}^*$

Comparison

$\varphi\colon S \to S$ and $(q^{\frac{1}{2}})^n = 1$ with n odd

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- a φ–invariant $\mathrm{SL}_2(\mathbb{C})$–character $r \in \mathcal{X}_{\mathrm{SL}_2}(S)$
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Suppose that the $\mathrm{PSL}_2(\mathbb{C})$*–character underlying* $\bar{r} \in \mathcal{X}^{\mathrm{enh}}_{\mathrm{PSL}_2(\mathbb{C})}(S)$ *is the reduction of the* $\mathrm{SL}_2(\mathbb{C})$*–character* $r \in \mathcal{X}_{\mathrm{SL}_2}(S)$*, and that* $p_v = h_v^{\frac{1}{2}} + h_v^{\frac{1}{2}}$ *for every puncture* v*. Then, there is an isomorphism* $V \to \bar{V}$ *conjugating* $\Lambda^q_{\varphi,r,p_v}$ *to a scalar multiple of* $\bar{\Lambda}^q_{\varphi,\bar{r},h_v}$

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Proof with a small lie.

Earlier work of FB + H. Wong $\Longrightarrow$ for every ideal triangulation τ, there is a quantum trace embedding $\mathcal{S}^q_{\mathrm{SL}_2}(S) \to \mathcal{CF}^q_\tau$ that is compatible with the Chekhov-Fock coordinate changes $\Psi^q_{\tau\tau'}$

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If $\bar{\Lambda}^q_{\varphi,\bar{r},h_v}$ conjugates $\bar{\rho}$ to its image under φ, it also conjugates ρ to its image under φ □

Comparison

The small lie. Because of the difference between $\mathrm{SL}_2(\mathbb{C})$ and $\mathrm{PSL}_2(\mathbb{C})$, these are a few sign issues to be resolved

Technical advantage of the quantum Teichmüller space. The representation theory is completely explicit

Computing $\bar{\Lambda}^q_{\varphi,\bar{r},h_v}$

To compute the isomorphism $\bar{\Lambda}^q_{\varphi,\bar{r},h_v}$, connect the ideal triangulation τ to $\varphi(\tau)$ by as sequence of ideal triangulations $\tau = \tau_0$, τ_1, τ_2, ..., $\tau_{k_0} = \varphi(\tau)$ where each τ_k is obtained from τ_{k+1}. Then:

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- these edge weights and the puncture weights h_v determine representations $\bar{\rho}_k\colon\mathcal{CF}^q_{\tau_k}\to\mathrm{End}(V_k)$ such that each $\bar{\rho}_k$ is isomorphic to $\bar{\rho}_{k-1}\circ\Psi^q_{\tau_{k-1}\tau_k}$ by an isomorphism $\bar{\Lambda}_k\colon V_k\to V_{k-1}$

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- can take $\bar{\Lambda}^q_{\varphi,\bar{r},h_v} = \bar{\Lambda}_1 \circ \bar{\Lambda}_2 \circ \cdots \circ \bar{\Lambda}_{k_0}$

Example: the one-puncture torus

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Fact. Every $\varphi \in \mathrm{SL}_2(\mathbb{Z})$ is conjugate to $\pm\varphi_1 \circ \varphi_2 \circ \cdots \circ \varphi_{k_0}$ where each φ_k is equal to $L = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ or $R = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$

Example: the one-puncture torus

$S =$ one-puncture torus $\qquad \pi_0 \operatorname{Diff}^+(S) = \mathrm{SL}_2(\mathbb{Z})$

Fact. Every $\varphi \in \mathrm{SL}_2(\mathbb{Z})$ is conjugate to $\pm\varphi_1 \circ \varphi_2 \circ \cdots \circ \varphi_{k_0}$ where each φ_k is equal to $L = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ or $R = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

Can assume $\varphi = \varphi_1 \circ \varphi_2 \circ \cdots \circ \varphi_{k_0}$

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Can assume $\varphi = \varphi_1 \circ \varphi_2 \circ \cdots \circ \varphi_{k_0}$

Get a sequence of ideal triangulations $\tau_0, \tau_1, \tau_2, \ldots, \tau_{k_0} = \varphi(\tau_0)$, with

$$\tau_k = \varphi_1 \circ \varphi_2 \circ \cdots \circ \varphi_k(\tau_0) \quad ,$$

such that each τ_k is obtained from τ_{k-1} by a diagonal exchange

Example: the one-puncture torus

The formulas involve the *Faddeev-Kashaev discrete quantum dilogarithm*

$$\mathrm{QDL}^q(u, v \mid i) = v^{-i} \prod_{j=1}^{i} (1 + uq^{-2j+1})$$

defined for q, u, $v \in \mathbb{C}$ and $i \in \mathbb{Z}$ with q^n and $v^n = 1 + u^n$

It is n–periodic, namely $\mathrm{QDL}^q(u, v \mid i + n) = \mathrm{QDL}^q(u, v \mid i)$

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Also,

$$\begin{aligned} D^q(u) &= \prod_{i=1}^{n} \mathrm{QDL}^q(u, v \mid i) \\ &= (1 + u^n)^{-\frac{n+1}{2}} \prod_{j=1}^{n} (1 + uq^{-2j+1})^{n-j+1}. \end{aligned}$$

Example: the one-puncture torus

$$\mathrm{Trace}\,\bar{\Lambda}^q_{\varphi,\bar{r},h_v} = \frac{1}{n^{\frac{k_0}{2}} \prod_{k=1}^{k_0} \left| D^q(\mathrm{e}^{\frac{1}{n}U_k}) \right|^{\frac{1}{n}}} \sum_{i_1,\, i_2, \dots,\, i_{k_0}=1}^{n} \prod_{k=1}^{k_0} \mathrm{QDL}^q(\mathrm{e}^{\frac{1}{n}U_k}, \mathrm{e}^{\frac{1}{n}V_k} \,|\, 2i_k) \, q^{\sum_{k=1}^{k_0} i_k^2(\varepsilon_k+\varepsilon_{k+1}+2) - 4\sum_{k=1}^{k_0} \varepsilon_{k+1} i_k i_{k+1}} \, q^{\varepsilon_1 l_0 i_1 + \frac{-\varepsilon_1 l_0 - m_0 + n_0}{2} i_{k_0}}$$

where $\varepsilon_k = \begin{cases} -1 & \text{if } \varphi_k = L \\ +1 & \text{if } \varphi_k = R. \end{cases}$, where the quantities U_k, $V_k \in \mathbb{C}$ are determined by careful choices of logarithms for the shearbend edge weights of $\bar{r} \in \mathcal{X}^{\mathrm{enh}}_{\mathrm{PSL}_2(\mathbb{C})}(S)$, and where l_0, m_0, $n_0 \in \mathbb{Z}$ are correction terms for the lack of periodicity of these logarithms

Part V

Analytic techniques

Example: the one-puncture torus

Recall. $S =$ one-puncture torus $\qquad \varphi \colon S \to S$

$\varphi \in \pi_0 \operatorname{Diff}^+(S) = \mathrm{SL}_2(\mathbb{Z})$

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$\varphi = \varphi_1 \circ \varphi_2 \circ \cdots \circ \varphi_{k_0}$ where each φ_k is equal to $L = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ or $R = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$

Example: the one-puncture torus

We are interested in the growth rate of the modulus of

$$\mathrm{Trace}\,\bar{\Lambda}^q_{\varphi,\bar{r},h_v} = \frac{1}{n^{\frac{k_0}{2}} \prod_{k=1}^{k_0} \left| D^q(\mathrm{e}^{\frac{1}{n}U_k}) \right|^{\frac{1}{n}}} \sum_{i_1,\, i_2, \ldots,\, i_{k_0}=1}^{n} \prod_{k=1}^{k_0} \mathrm{QDL}^q(\mathrm{e}^{\frac{1}{n}U_k}, \mathrm{e}^{\frac{1}{n}V_k} \,|\, 2i_k) \, q^{\sum_{k=1}^{k_0} i_k^2(\varepsilon_k+\varepsilon_{k+1}+2) - 4\sum_{k=1}^{k_0} \varepsilon_{k+1} i_k i_{k+1}} \, q^{\varepsilon_1 l_0 i_1 + \frac{-\varepsilon_1 l_0 - m_0 + n_0}{2} i_{k_0}}$$

where $\varepsilon_k = \begin{cases} -1 & \text{if } \varphi_k = L \\ +1 & \text{if } \varphi_k = R. \end{cases}$, where the quantities $U_k,\ V_k \in \mathbb{C}$, with $\mathrm{e}^{V_k} = 1 + \mathrm{e}^{U_k}$, are determined by careful choices of logarithms for the shearbend edge weights of $\bar{r} \in \mathcal{X}^{\mathrm{enh}}_{\mathrm{PSL}_2(\mathbb{C})}(S)$, and where $l_0,\ m_0,\ n_0 \in \mathbb{Z}$ are correction terms for the lack of periodicity of these logarithms

Example: the one-puncture torus

QDL^q is the Faddeev-Kashaev discrete quantum dilogarithm

$$\mathrm{QDL}^q(u, v \mid i) = v^{-i} \prod_{j=1}^{i} (1 + uq^{-2j+1})$$

defined for u, $v \in \mathbb{C}$ with $v^n = 1 + u^n$ and $i \in \mathbb{Z}$

It is n–periodic, namely $\mathrm{QDL}^q(u, v \mid i + n) = \mathrm{QDL}^q(u, v \mid i)$

Also,

$$D^q(u) = \prod_{i=1}^{n} \mathrm{QDL}^q(u, v \mid i)$$

$$= (1 + u^n)^{-\frac{n+1}{2}} \prod_{j=1}^{n} (1 + uq^{-2j+1})^{n-j+1}.$$

Example: the one-puncture torus

Proposition

Let $U \in \mathbb{C}$ be given, with $\mathrm{e}^{U} \neq -1$. For every odd n, set $q = \mathrm{e}^{\frac{2\pi \mathrm{i}}{n}}$. Then,

$$\lim_{\substack{n \to \infty \\ n = 1 \bmod 4}} \left| D^q(\mathrm{e}^{\frac{1}{n} U}) \right|^{\frac{1}{n}} = 2^{\frac{\operatorname{Im} U}{4\pi}} \left| \frac{\cosh \frac{U + \pi \mathrm{i}}{4}}{\cosh \frac{U - \pi \mathrm{i}}{4}} \right|^{\frac{1}{4}}$$

$$\lim_{\substack{n \to \infty \\ n = 3 \bmod 4}} \left| D^q(\mathrm{e}^{\frac{1}{n} U}) \right|^{\frac{1}{n}} = 2^{\frac{\operatorname{Im} U}{4\pi}} \left| \frac{\sinh \frac{U + \pi \mathrm{i}}{4}}{\sinh \frac{U - \pi \mathrm{i}}{4}} \right|^{\frac{1}{4}}.$$

Proof.

Undergraduate math + brute force □

Example: the one-puncture torus

Therefore, we only need to understand the asymptotics of the sum

$$S_n = \sum_{i_1, i_2, \ldots, i_{k_0}=1}^{n} \prod_{k=1}^{k_0} \mathrm{QDL}^q(\mathrm{e}^{\frac{1}{n}U_k}, \mathrm{e}^{\frac{1}{n}V_k} \mid 2i_k)$$

$$q^{\sum_{k=1}^{k_0} i_k^2(\varepsilon_k+\varepsilon_{k+1}+2)-4\sum_{k=1}^{k_0}\varepsilon_{k+1} i_k i_{k+1}}$$

$$q^{\varepsilon_1 l_0 i_1 + \frac{-\varepsilon_1 l_0 - m_0 + n_0}{2} i_{k_0}}$$

The quantum dilogarithms

Recall that, for $q^n = 1$, the *discrete quantum dilogarithm* of Faddeev-Kashaev is

$$\mathrm{QDL}^q(u, v \mid i) = v^{-i} \prod_{j=1}^{i} (1 + uq^{-2j+1})$$

defined for u, $v \in \mathbb{C}$ with $v^n = 1 + u^n$ and $i \in \mathbb{Z}$

The quantum dilogarithms

For $\hbar > 0$ and $z \in \mathbb{C}$ with $-\frac{\pi\hbar}{2} < \operatorname{Re} z < \pi + \frac{\pi\hbar}{2}$, the *small continuous quantum dilogarithm* of Faddeev is

$$\mathrm{li}_2^{\hbar}(z) = \frac{12z^2 - 12\pi z + 2\pi^2 - \pi^2\hbar^2}{12} + 2\pi \mathrm{i}\hbar \int_0^{+\infty} \left(\frac{\sinh(2z-\pi)t}{2t\sinh(\pi t)\sinh(\pi\hbar t)} - \frac{2z-\pi}{2\pi^2\hbar t^2} \right) dt.$$

where the integrand of the integral continuously extends to $[0, +\infty[$ by taking the value $\frac{(2z-\pi)(4z^2-4\pi z-\pi^2\hbar^2)}{12\pi^2\hbar}$ at $t = 0$.

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The *big continuous quantum dilogarithm* is

$$\mathrm{Li}_2^{\hbar}(z) = \mathrm{e}^{\frac{1}{2\pi\mathrm{i}\hbar}\mathrm{li}_2^{\hbar}(z)}$$

The quantum dilogarithms

Proposition

The big quantum dilogarithm function $\mathrm{Li}_2^{\hbar}(z)$ *has a unique meromorphic extension to the plane* $\mathbb{C}$*, with poles all contained in* $]-\infty, 0[$ *and zeros all contained in* $]\pi, \infty[$*, such that*

$$\mathrm{Li}_2^{\hbar}(z + \pi\hbar) = (1 - \mathrm{e}^{2\mathrm{i}z + \pi\mathrm{i}\hbar})^{-1}\mathrm{Li}_2^{\hbar}(z)$$

Corollary

If $q = \mathrm{e}^{\frac{2\pi\mathrm{i}}{n}}$, $\hbar = \frac{2}{n}$ *and* $u = \mathrm{e}^{\frac{1}{n}U}$ *then*

$$\mathrm{QDL}^q(u, v \,|\, j) = \mathrm{e}^{-\frac{j}{n}V}\frac{\mathrm{Li}_2^{\frac{2}{n}}\left(\frac{\pi}{2} - \frac{\pi}{n} + \frac{1}{2n\mathrm{i}}U - \frac{2\pi j}{n}\right)}{\mathrm{Li}_2^{\frac{2}{n}}\left(\frac{\pi}{2} - \frac{\pi}{n} + \frac{1}{2n\mathrm{i}}U\right)}$$

The sum in analytic form

Recall. We are interested in the asymptotics of

$$S_n = \sum_{i_1, i_2, \ldots, i_{k_0}=1}^{n} \prod_{k=1}^{k_0} \mathrm{QDL}^q(\mathrm{e}^{\frac{1}{n}U_k}, \mathrm{e}^{\frac{1}{n}V_k} \mid 2i_k)$$

$$q^{\sum_{k=1}^{k_0} i_k^2(\varepsilon_k+\varepsilon_{k+1}+2)-4\sum_{k=1}^{k_0}\varepsilon_{k+1} i_k i_{k+1}}$$

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$$q^{\varepsilon_1 l_0 i_1 + \frac{-\varepsilon_1 l_0 - m_0 + n_0}{2} i_{k_0}}$$

$$= \sum_{i_1, i_2, \dots, i_{k_0}=1}^{n} g\left(\tfrac{2\pi i_1}{n}, \tfrac{2\pi i_1}{n}, \dots, \tfrac{2\pi i_{k_0}}{n}\right) \exp\left(\tfrac{n}{4\pi \mathrm{i}} f_n\left(\tfrac{2\pi i_1}{n}, \tfrac{2\pi i_1}{n}, \dots, \tfrac{2\pi i_{k_0}}{n}\right)\right)$$

The sum in analytic form

with

$$f_n(\alpha_1,\alpha_2,\ldots,\alpha_{k_0}) = \sum_{k=1}^{k_0} \mathrm{li}_2^{\frac{2}{n}}\left(\frac{\pi}{2} - \frac{\pi}{n} + \frac{1}{2n\mathrm{i}} U_k - 2\alpha_k\right)$$

$$- \sum_{k=1}^{k_0} \mathrm{li}_2^{\frac{2}{n}}\left(\frac{\pi}{2} - \frac{\pi}{n} + \frac{1}{2n\mathrm{i}} U_k\right) - 2\sum_{k=1}^{k_0}(\varepsilon_k + \varepsilon_{k+1} + 2)\alpha_k^2$$

$$+ 8\sum_{k=1}^{k_0} \varepsilon_{k+1}\alpha_k\alpha_{k+1}$$

$$g(\alpha_1,\alpha_2,\ldots,\alpha_{k_0}) = \prod_{k=1}^{k_0} \mathrm{e}^{-\frac{\alpha_k}{\pi} V_k} \exp\left(\varepsilon_1 \widehat{l_0}\alpha_1\mathrm{i} + \frac{-\varepsilon_1\widehat{l_0} - \widehat{m}_0 + \widehat{n}_0}{2}\alpha_{k_0}\mathrm{i}\right)$$

and since $q = \mathrm{e}^{\frac{2\pi\mathrm{i}}{n}}$

The sum in analytic form

Proposition

For every z with $0 < \mathrm{Re}\, z < \pi$ as $\hbar \to 0$

$$\mathrm{li}_2^{\frac{2}{n}}(z) = \mathrm{li}_2(\mathrm{e}^{2\mathrm{i}z}) + O(\tfrac{1}{n^2})$$

where li_2 is the classical dilogarithm

$$\mathrm{li}_2(u) = -\int_0^u \frac{\log(1-t)}{t}\, dt.$$

In addition, the convergence is uniform on compact subsets of the strip $\{z \in \mathbb{C}; 0 < \mathrm{Re}\, z < \pi\}$

The sum in analytic form

Therefore, as $n \to \infty$, $f_n(\alpha_1, \alpha_2, \ldots, \alpha_{k_0}) \to f_\infty(\alpha_1, \alpha_2, \ldots, \alpha_{k_0})$ with

$$f_\infty(\alpha_1, \alpha_2, \ldots, \alpha_{k_0}) = \sum_{k=1}^{k_0} \mathrm{li}_2\left(-\mathrm{e}^{-4\mathrm{i}\alpha_k}\right) + k_0 \tfrac{\pi^2}{12} - 4\sum_{k=1}^{k_0} \tfrac{\varepsilon_k + \varepsilon_{k+1} + 2}{2} \alpha_k^2 + 8 \sum_{k=1}^{k_0} \varepsilon_{k+1} \alpha_k \alpha_{k+1}$$

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for some quadratic function $Q(\alpha_1, \alpha_2, \ldots, \alpha_{k_0})$

Approximate discrete sum by integral

We want the asymptotics of

$$S_n = \sum_{i_1,\, i_2, \ldots,\, i_{k_0}=1}^{n} g\left(\frac{2\pi i_1}{n}, \frac{2\pi i_1}{n}, \ldots, \frac{2\pi i_{k_0}}{n}\right) \exp\left(\frac{n}{4\pi \mathrm{i}} f_n\left(\frac{2\pi i_1}{n}, \frac{2\pi i_1}{n}, \ldots, \frac{2\pi i_{k_0}}{n}\right)\right)$$

with $f_n(\alpha_1, \alpha_2, \ldots, \alpha_{k_0}) \to f_\infty(\alpha_1, \alpha_2, \ldots, \alpha_{k_0})$

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with $f_n(\alpha_1, \alpha_2, \ldots, \alpha_{k_0}) \to f_\infty(\alpha_1, \alpha_2, \ldots, \alpha_{k_0})$

Step 1. Riemann sum approximation

$$S_n \approx \left(\tfrac{n}{2\pi}\right)^{k_0} \int_{[0,2\pi]^{k_0}} g(\alpha_1, \alpha_2, \ldots, \alpha_{k_0}) \exp\left(\tfrac{n}{4\pi \mathrm{i}} f_\infty(\alpha_1, \alpha_2, \ldots, \alpha_{k_0})\right) d\alpha_1\, d\alpha_2\, \ldots\, d\alpha_{k_0}$$

The stationary phase method

Step 2. Well-known principle in mathematics/physics

$$S_n \approx \left(\tfrac{n}{2\pi}\right)^{k_0} \int_{[0,2\pi]^{k_0}} g(\alpha_1, \alpha_2, \ldots, \alpha_{k_0}) \exp\left(\tfrac{n}{4\pi \mathrm{i}} f_\infty(\alpha_1, \alpha_2, \ldots, \alpha_{k_0})\right) d\alpha_1\, d\alpha_2\, \ldots\, d\alpha_{k_0}$$

$$\approx \left(\tfrac{n}{2\pi}\right)^{k_0} \frac{\text{constant}}{n^{\frac{k_0}{2}}}\, g(c) \exp\left(\tfrac{n}{4\pi \mathrm{i}} f_\infty(c)\right)$$

for some complex critical point c of f_∞

Search for critical points

The search for a complex critical point of

$$f_\infty(\alpha_1, \alpha_2, \ldots, \alpha_{k_0}) = \sum_{k=1}^{k_0} \mathrm{li}_2\left(-\mathrm{e}^{-4\mathrm{i}\alpha_k}\right) + Q(\alpha_1, \alpha_2, \ldots, \alpha_{k_0})$$

is very similar to classical techniques (Casson, Rivin, Neumann-Zagier, Yoshida) to explicitly find the hyperbolic metric on the mapping torus M_φ

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is very similar to classical techniques (Casson, Rivin, Neumann-Zagier, Yoshida) to explicitly find the hyperbolic metric on the mapping torus M_φ, and will give

$$|S_n| \approx \text{constant}\, n^{\frac{k_0}{2}} \exp\left(\frac{n}{4\pi}\,\mathrm{vol}_{\mathrm{hyp}}\, M_\varphi\right)$$

which is what we wanted

Except

Except

This is all wrong!!

Approximate discrete sum by integral

Step 1. Riemann sum approximation

$$S_n = \sum_{i_1, i_2, \ldots, i_{k_0}=1}^{n} g\left(\frac{2\pi i_1}{n}, \frac{2\pi i_1}{n}, \ldots, \frac{2\pi i_{k_0}}{n}\right) \exp\left(\frac{n}{4\pi \mathrm{i}} f_n\left(\frac{2\pi i_1}{n}, \frac{2\pi i_1}{n}, \ldots, \frac{2\pi i_{k_0}}{n}\right)\right)$$

$$\approx \left(\frac{n}{2\pi}\right)^{k_0} \int_{[0,2\pi]^{k_0}} g(\alpha_1, \alpha_2, \ldots, \alpha_{k_0}) \exp\left(\frac{n}{4\pi \mathrm{i}} f_\infty(\alpha_1, \alpha_2, \ldots, \alpha_{k_0})\right) d\alpha_1\, d\alpha_2 \ldots d\alpha_{k_0}$$

since $f_n(\alpha_1, \alpha_2, \ldots, \alpha_{k_0}) \to f_\infty(\alpha_1, \alpha_2, \ldots, \alpha_{k_0})$

Approximate discrete sum by integral

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since $f_n(\alpha_1, \alpha_2, \ldots, \alpha_{k_0}) \to f_\infty(\alpha_1, \alpha_2, \ldots, \alpha_{k_0})$

What about error approximation?

Approximate discrete sum by integral

Step 1. Riemann sum approximation

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$$\approx \left(\frac{n}{2\pi}\right)^{k_0} \int_{[0,2\pi]^{k_0}} g(\alpha_1, \alpha_2, \ldots, \alpha_{k_0}) \exp\left(\frac{n}{4\pi \mathrm{i}} f_\infty(\alpha_1, \alpha_2, \ldots, \alpha_{k_0})\right) d\alpha_1\, d\alpha_2 \ldots d\alpha_{k_0}$$

since $f_n(\alpha_1, \alpha_2, \ldots, \alpha_{k_0}) \to f_\infty(\alpha_1, \alpha_2, \ldots, \alpha_{k_0})$

What about error approximation?

$f_\infty(\alpha_1, \alpha_2, \ldots, \alpha_{k_0}) = \sum_{k=1}^{k_0} \mathrm{li}_2\left(-\mathrm{e}^{-4\mathrm{i}\alpha_k}\right) + Q(\alpha_1, \alpha_2, \ldots, \alpha_{k_0})$
for $Q(\alpha_1, \alpha_2, \ldots, \alpha_{k_0})$ quadratic

Toy model

$$S_n = \sum_{j=1}^{n} \exp\left(\frac{n}{4\pi \mathrm{i}} f\left(\frac{2\pi j}{n}\right)\right) \text{ with } f(\alpha) = \mathrm{li}_2\left(\mathrm{e}^{\mathrm{i}\alpha}\right) + 2\alpha^2$$

Toy model

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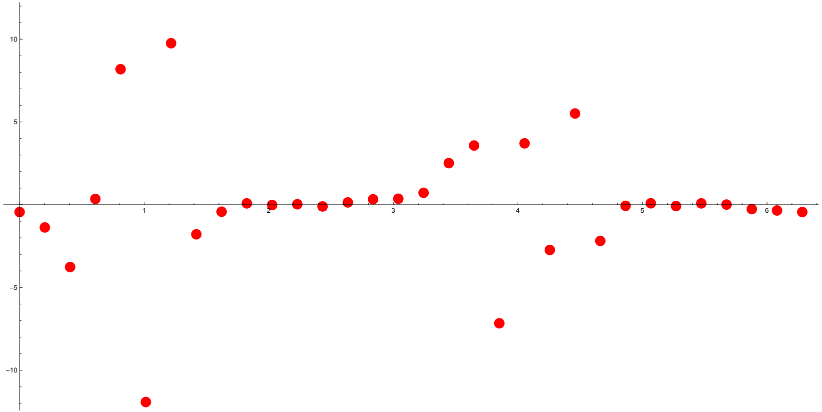
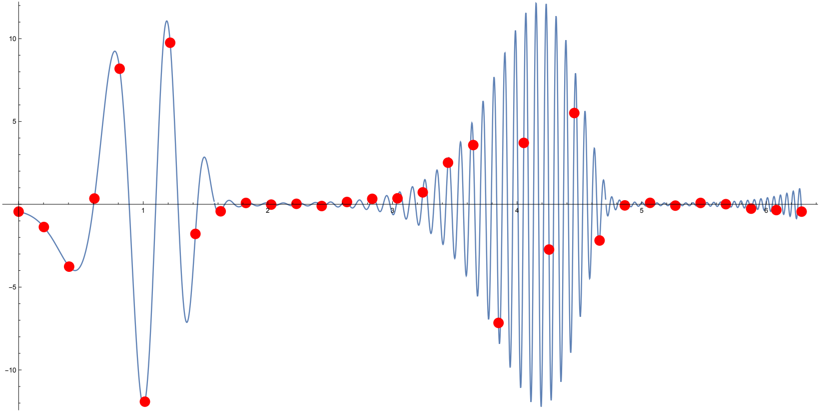
Plot the terms $\exp\left(\frac{n}{4\pi \mathrm{i}} f\left(\frac{2\pi j}{n}\right)\right)$



Toy model

$$S_n = \sum_{j=1}^{n} \exp\left(\frac{n}{4\pi i} f\left(\frac{2\pi j}{n}\right)\right) \text{ with } f(\alpha) = \mathrm{li}_2\left(e^{i\alpha}\right) + 2\alpha^2$$

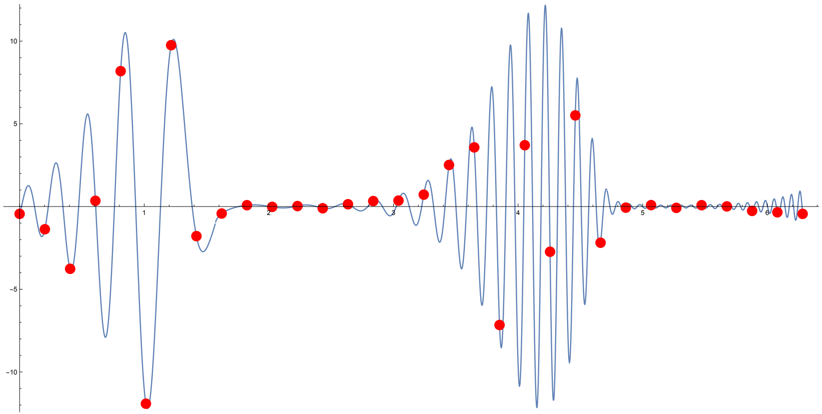
Plot the terms $\exp\left(\frac{n}{4\pi i} f\left(\frac{2\pi j}{n}\right)\right)$ and the function $\exp\left(\frac{n}{4\pi i} f(\alpha)\right)$



Toy model

$$S_n = \sum_{j=1}^{n} \exp\left(\frac{n}{4\pi \mathrm{i}} f\left(\frac{2\pi j}{n}\right)\right) \text{ with } f(\alpha) = \mathrm{li}_2\left(\mathrm{e}^{\mathrm{i}\alpha}\right) + 2\alpha^2$$

Plot the terms $\exp\left(\frac{n}{4\pi \mathrm{i}} f\left(\frac{2\pi j}{n}\right)\right)$ and the function $\exp\left(\frac{n}{4\pi \mathrm{i}} f(\alpha)\right) \exp(-n\alpha \mathrm{i})$



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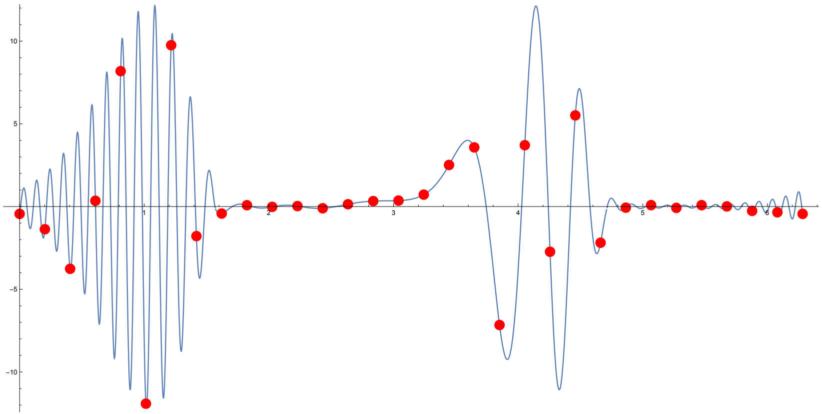

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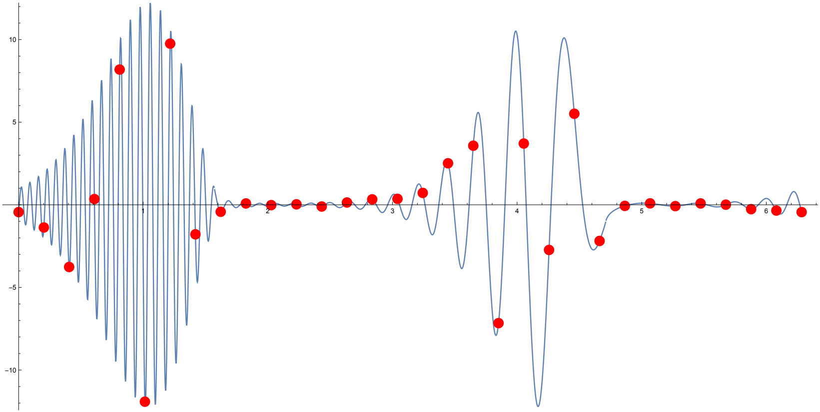

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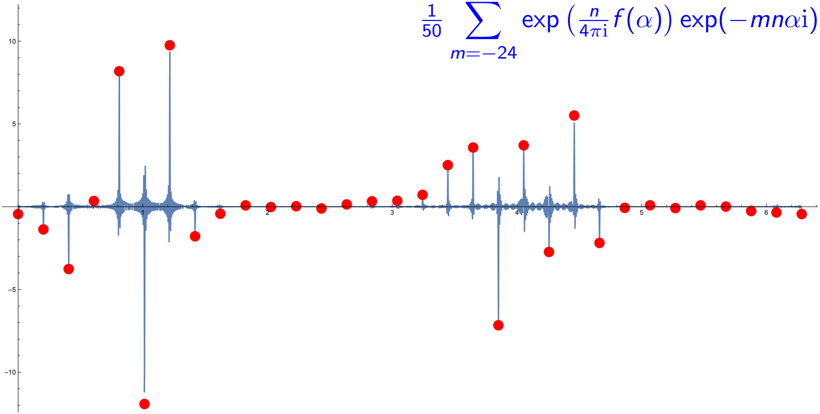
$$\frac{1}{50}\sum_{m=-24}^{25} \exp\left(\frac{n}{4\pi \mathrm{i}} f(\alpha)\right) \exp(-mn\alpha \mathrm{i})$$



The Poisson Summation Formula

If $h\colon \mathbb{R} \to \mathbb{R}$ is 2π–periodic, continuous, and a little regular

$$\sum_{j=1}^{n} h\left(\frac{2\pi j}{n}\right) = n \sum_{m=-\infty}^{\infty} \widehat{h}(m\,n) = \frac{n}{2\pi} \sum_{m=-\infty}^{\infty} \int_0^{2\pi} h(\alpha)\mathrm{e}^{-mn\alpha\mathrm{i}}\, d\alpha$$

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Problem

The function $h(\alpha) = \exp\left(\frac{n}{4\pi\mathrm{i}} f(\alpha)\right) = \exp\left(\frac{n}{4\pi\mathrm{i}} \mathrm{li}_2\left(\mathrm{e}^{\mathrm{i}\alpha}\right) + \frac{n\alpha^2}{2\pi\mathrm{i}}\right)$ is not 2π–periodic, only at the points of the form $\alpha = \frac{2\pi j}{n}$

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Problem and solution

The function $h(\alpha) = \exp\left(\frac{n}{4\pi \mathrm{i}} f(\alpha)\right) = \exp\left(\frac{n}{4\pi \mathrm{i}} \mathrm{li}_2\left(\mathrm{e}^{\mathrm{i}\alpha}\right) + \frac{n\alpha^2}{2\pi \mathrm{i}}\right)$ is not 2π–periodic, only at the points of the form $\alpha = \frac{2\pi j}{n}$

Need to introduce a *Twisted Poisson Summation Formula*

The Poisson summation formula

Method pioneered by Ohtsuki (and D. Thurston)

$$S_n = \sum_{i_1,\, i_2, \dots,\, i_{k_0}=1}^{n} g\left(\tfrac{2\pi i_1}{n}, \tfrac{2\pi i_2}{n}, \dots, \tfrac{2\pi i_{k_0}}{n}\right) \exp\left(\tfrac{n}{4\pi \mathrm{i}} f_n\left(\tfrac{2\pi i_1}{n}, \tfrac{2\pi i_2}{n}, \dots, \tfrac{2\pi i_{k_0}}{n}\right)\right)$$

$$= \left(\tfrac{n}{2\pi}\right)^{k_0} \sum_{m_1,\, m_2, \dots,\, m_{k_0}=-\infty}^{\infty} \widehat{F}_n(m_1, m_2, \dots, m_{k_0})$$

with

$$\widehat{F}_n(m_1, m_2, \dots, m_{k_0}) = \int_{[0,2\pi]^{k_0}} g_n(\alpha_1, \alpha_2, \dots, \alpha_{k_0}) \exp\left(\tfrac{n}{4\pi \mathrm{i}}\left(f_n(\alpha_1, \alpha_2, \dots, \alpha_{k_0}) + \sum_{k=1}^{k_0} m_k \pi \alpha_k\right)\right) d\alpha_1\, d\alpha_2 \dots d\alpha_{k_0}$$

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Idea: a handful of these integrals will dominate all the other ones

The saddle point method

To estimate

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Because of the explicit form of
$f_\infty(\alpha_1, \alpha_2, \ldots, \alpha_{k_0}) = \sum_{k=1}^{k_0} \mathrm{li}_2\left(-\mathrm{e}^{-4\mathrm{i}\alpha_k}\right) + Q(\alpha_1, \alpha_2, \ldots, \alpha_{k_0})$
this can be done "by hand"

Connections with hyperbolic geometry

The quadratic term

$$Q(\alpha_1, \alpha_2, \ldots, \alpha_{k_0}) = -4 \sum_{k=1}^{k_0} \tfrac{\varepsilon_k + \varepsilon_{k+1} + 2}{2} \alpha_k^2 + 8 \sum_{k=1}^{k_0} \varepsilon_{k+1} \alpha_k \alpha_{k+1}$$

is determined by the decomposition of the diffeomorphism $\varphi\colon S \to S$ as

$$\varphi = \varphi_1 \circ \varphi_2 \circ \cdots \circ \varphi_{k_0}$$

where each φ_k is equal to $L = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ or $R = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$

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This combinatorial data also gave us a sequence of ideal triangulations τ_0, τ_1, τ_2, ..., $\tau_{k_0} = \varphi(\tau_0)$, which gives us an ideal triangulations of the mapping torus M_φ, namely a decomposition of M_φ into ideal tetrahedra

Connections with hyperbolic geometry

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The Geometric Hypothesis is widely assumed to always hold. It can be effectively checked by computer on specific examples.

Connections with hyperbolic geometry

Assuming that the Geometric Hypothesis holds, exactly 4^{k_0} critical points of the function

$$f_n(\alpha_1, \alpha_2, \ldots, \alpha_{k_0}) + \sum_{k=1}^{k_0} m_k \pi \alpha_k$$

contribute leading terms $\asymp n^{\frac{k_0}{2}} \exp\left(\frac{n}{4\pi} \operatorname{vol}_{\mathrm{hyp}} M_\varphi\right)$ to the integrals

$$\widehat{F}_n(m_1, m_2, \ldots, m_{k_0}) = \int_{[0,2\pi]^{k_0}} g_n(\alpha_1, \alpha_2, \ldots, \alpha_{k_0}) \exp\left(\frac{n}{4\pi \mathrm{i}}\left(f_n(\alpha_1, \alpha_2, \ldots, \alpha_{k_0}) + \sum_{k=1}^{k_0} m_k \pi \alpha_k\right)\right) d\alpha_1\, d\alpha_2\, \ldots\, d\alpha_{k_0}$$

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In particular, the leading terms will all cancel out for the growth rate of $\mathrm{Trace}\, \bar\Lambda^q_{\varphi,\bar r,h_v}$ in quantum Teichmüller theory, when the φ–invariant $\mathrm{PSL}_2(\mathbb{C})$–character $\bar r \in \mathcal{X}^{\mathrm{enh}}_{\mathrm{PSL}_2(\mathbb{C})}(S)$ does not lift to a φ–invariant $\mathrm{SL}_2(\mathbb{C})$–character $r \in \mathcal{X}_{\mathrm{SL}_2}(S)$

Thank you