Colloquium Cergy-Pontoise
An introduction to quantum topology

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December 9, 2024

Plan

I will try to provide a general overview of some of the key ideas and definitions in "quantum topology". At the end of the talk, I will try to hint at how these ideas evolved and how they are related to the talks of the next Skein days...

1. Recalls on manifolds and cobordisms
2. Topological Quantum Field Theories
3. Quantization
4. Skein algebras as quantizations
5. More recent developments

What is a quantum field theory ?

Very roughly in theoretical physics a temptative definition of a quantum field theory is a way to associate:

1. a Hilbert space ("state space") to each "space" (e.g. riemannian manifold, possibly endowed with some bundle..)
2. a unitary transformation to each "evolution" ("spacetime") between two spaces

Unfortunately currently such a temptative definition cannot be formalised at this level of generality.
In this talk I will formalise the notion of “topological quantum field theory", and for this I will start by the notion of manifolds and cobordisms...

Manifolds and orientations

Let me recall that a n-manifold M is a topological space locally homeomorphic to $\mathbb{R}^n$ (Haussdorf, countable base).
A "manifold with boundary" is a topological space locally homeomorphic to $\mathbb{R}^n$ or to the half space $\mathbb{R}^n_{x_1 \geq 0}$.
All manifolds will be compact, smooth and oriented.
An orientation is a "coherent choice of the notion of "positive" tangent basis at all points of M".

Example

The interval $[0, 1]$ is a 1-manifold with boundary $\{0, 1\}$. It has two orientations.
A circle is a 1-manifold without boundary, it has 2 orientations.
A disc $D = \{z \in \mathbb{C} | |z| \leq 1\}$ is an orientable 2-manifold with boundary a circle. It has two orientations.
A sphere $S^2 = \{\vec{x} \in \mathbb{R}^3 | ||\vec{x}|| = 1\}$ is a 2-manifold without boundary. It has 2 orientations.

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Example

An orientation on a 0-manifold (i.e. a discrete set of points) is the assignment of a sign $\pm$ to each point.

From now on all manifolds will be oriented unless stated the contrary.

Cobordisms

Definition (Cobordism)

If M_1, M_2 are n-manifolds, a cobordism W from M_1 to M_2 is a $n+1$-manifold whose boundary is identified with $(-M_1) \sqcup M_2$.

So a cobordism is a manifold with the additional structure of a choice of how to identify its boundary as a disjoint union of two pieces...

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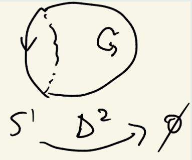
Example

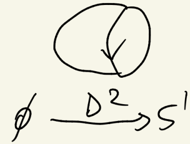
The interval $I = [0,1]$ oriented from left to right is a 1-manifold with boundary $\{-0, +1\}$.
As such it underlies a cobordism from $\{0\}$ to $\{1\}$.
But also one from $\emptyset$ to $\{-0\} \sqcup \{+1\}$.
Or finally one from $\{0\} \sqcup \{-1\}$ to $\emptyset$.
$\implies$ How I can be seen as a cobordism depends on how one chooses to orient I and to decompose $\partial I = \{-0, 1\}$. This additional structure is included in the datum of a cobordism.

Example

A disc $D^2 = \{z \in \mathbb{C} | |z| \leq 1\}$ is a 2-manifold with boundary. As such it can be seen as a cobordism from S^1 to $\emptyset$.



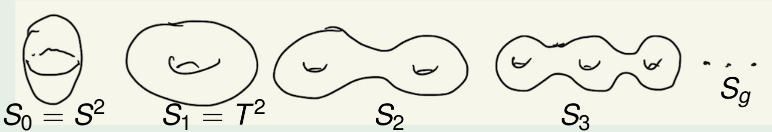
Or reciprocally as one from $\emptyset$ to S^1.



Cobordisms

Example

A sphere $S^2 = \{\vec{x} \in \mathbb{R}^3 | |\vec{x}| = 1\}$ is a compact 2-manifold without boundary: “a surface”.
More in general each compact, connected, oriented surface is diffeomorphic to one of the following list:



Each surface S_g is the boundary of a corresponding 3-manifold known as “handlebody” H_g which can then be seen as a cobordism from $\emptyset$ to S_g (or the other way round).

What is a Topological Quantum Field Theory (TQFT) ?

Definition (TQFT)

A $n+1$-TQFT is a way to associate:

1. *a $\mathbb{C}$-vector space $Z(M)$ to each n-dimensional compact, oriented manifold M without boundary.*
2. *a linear map $Z(W) : Z(M_1) \to Z(M_2)$ to each cobordism W (considered up to diffeomorphism) from M_1 to M_2.*

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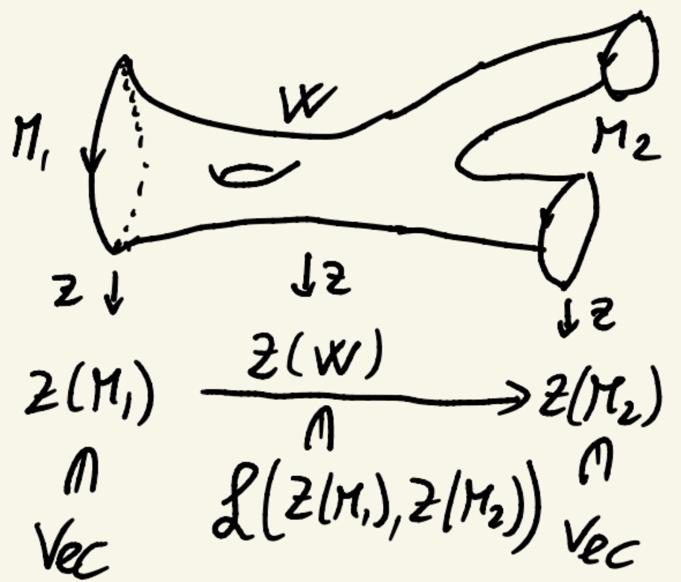

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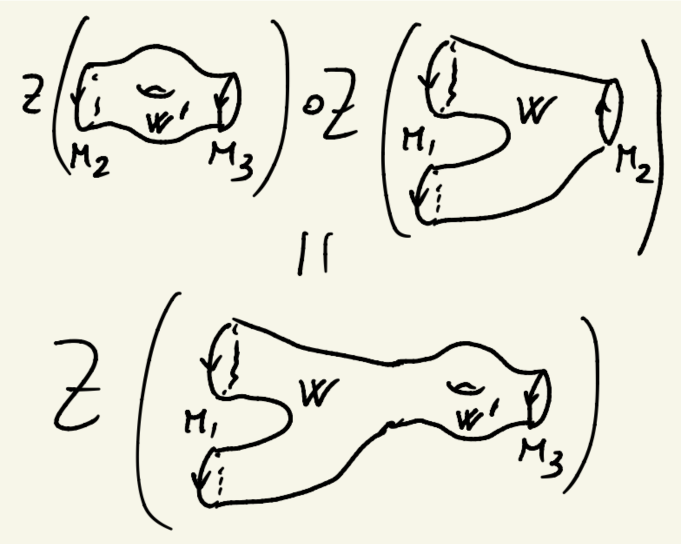
Such that :

1. *The linear maps "compose well" : if $W : M_1 \to M_2$ and $W' : M_2 \to M_3$ are cobordisms then*
$$Z(W' \circ W) = Z(W') \circ Z(W).$$
2. *Z is "multiplicative" ("monoidal" is the correct term) :*
$$Z(M_1 \sqcup M_2) = Z(M_1) \otimes Z(M_2) \text{ and } Z(W_1 \sqcup W_2) = Z(W_1) \otimes Z(W_2)$$
for all n-manifolds M_1, M_2 and all cobordisms W_1, W_2.

Graphical explanation: what is a TQFT?



In a few minutes

Example: a $1+1$-TQFT

A $1+1$ TQFT associates to each compact oriented 1-manifold (i.e. a set of circles) a vector space. By multiplicativity : if $Z(S^1) = A$ then

$$Z(S^1 \sqcup \cdots \sqcup S^1) = A \otimes A \otimes \cdots \otimes A \qquad \text{and} \;\; Z(\emptyset) = \mathbb{C}$$

Example: a $1 + 1$-TQFT

As we saw, a disc D^2 can be considered both as a cobordism which we denote $\mathbf{1}$ from $\emptyset \to S^1$ thus inducing

$$Z(\mathbf{1}) : Z(\emptyset) = \mathbb{C} \to A = Z(S^1)$$

and as a cobordism which we denote ϵ from $S^1 \to \emptyset$ thus inducing:

$$Z(\epsilon) : Z(S^1) = A \to \mathbb{C} = Z(\emptyset) = \mathbb{C}.$$

Example: a $1+1$-TQFT

The annulus $S^1 \times [-1,1]$ can be seen as a cobordism $Ann : S^1 \to S^1$ thus inducing:

$$Z(Ann) : A \to A \text{ (spoiler : it is the identity)}$$

Or we can see it as a cobordism from $S^1 \sqcup S^1$ to $\emptyset$, thus inducing:

$$Z(Cap) : A \otimes A \to \mathbb{C} \text{ (A bilinear form on A)}$$

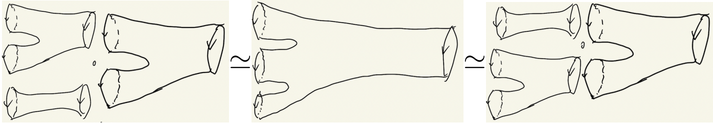
Exercice: this form is necessarily symmetric and non degenerate !

Example: a $1+1$-TQFT

The pant can be considered as a cobordism $P : S^1 \sqcup S^1 \to S^1$ thus inducing:

$$Z(P) : A \otimes A \to A$$

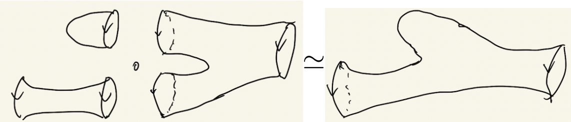
If one composes $P \circ (P \sqcup Ann)$ he gets a "tri-pant" which is diffeomorphic also to $P \circ (Ann \sqcup P)$:



Therefore $Z(P)$ is an associative product on $A = Z(S^1)$!

Example: a $1+1$-TQFT

Since $P \circ (D^2 \sqcup Ann) = Ann$ then $Z(D^2) \in A$ is the unit of this product:



Example: a $1+1$-TQFT

Overall we see that the vector space A is naturally endowed with a lot of algebraic structures, induced directly from the topology of surfaces. Indeed it is a Frobenius algebra :

Definition (Frobenius algebras)

A finite dimensional commutative $\mathbb{C}$-algebra A is Frobenius if it is endowed with a linear map $\epsilon : A \to \mathbb{C}$ such that the biliinear form $\langle a, b\rangle := \epsilon(ab)$ is non-degenerate.

And one can prove the following:

Theorem (Dijkgraaf? Abrams?, well detailed by Kock)

$1+1$ *TQFTs are in bijection with finite dimensional commutative Frobenius algebras.*

Why care ?

We already remarked that monoidality implies that $Z(\emptyset) = \mathbb{C}$:

$$(Indeed\ Z(\emptyset \sqcup N) = Z(\emptyset) \otimes Z(N) \implies Z(\emptyset) = \mathbb{C}.)$$

So if W is a $n+1$-manifold with empty boundary, then it can be seen as a cobordism from $\emptyset$ to $\emptyset$.
Hence, if we are given a TQFT Z and we apply it to W we get :

$$Z(W) \in \mathcal{L}(Z(\emptyset), Z(\emptyset)) = \mathcal{L}(\mathbb{C}, \mathbb{C}) = \mathbb{C}.$$

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$$Z(W) \in \mathcal{L}(Z(\emptyset), Z(\emptyset)) = \mathcal{L}(\mathbb{C}, \mathbb{C}) = \mathbb{C}.$$

Since by hypothesis $Z(W)$ depends on W only up to diffeomorphism $Z(W) \in \mathbb{C}$ is a invariant of W: this is what people call **a "quantum" invariant.**

Why care ?

Furthermore one can check that for each n-manifold, the group $Diff^+(M)$ acts on the vector space $Z(M)$.
Therefore one immediately has an action of $Diff^+(M)$ on $Z(M)$, this is what people called a "**quantum representation**".
(FACT: It actually turns out that it factors through the "mapping class group of M": $\pi_0(Diff^+(M))$.)

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(FACT: It actually turns out that it factors through the "mapping class group of M": $\pi_0(Diff^+(M))$.)
In particular when $n = 2$, one gets finite dimensional representations of the "mapping class group" of surfaces.
These groups are especially interesting for topologists and geometers and still are quite mysterious.
For instance : it is not yet known if they are linear, i.e. if they can be realised as subgroups of $GL_n(\mathbb{C})$ for some n big enough.

Witten-Reshetikhin-Turaev $2+1$ TQFTs

Finding interesting examples of $2+1$-TQFTs is non trivial. One of the many crucial contributions of E. Witten was his discovery of an infinite family of $2+1$ TQFTs, physically motivated. The construction of these TQFTs was later formalised (using completely different techniques based on quantum groups and their representations) by Reshetikhin and Turaev. Witten was awarded the Fields Medal in 1990 together with Vaughan Jones and Vladimir Drinfeld, for a list of results which are intimately connected and form a an extremely rich corpus ranging from knot theory, to quantum algebra, to mathematical physics... The consequences are still being studied today.
His construction takes as input a compact simple Lie group G (here $SU(2)$), an integer $k \geq 2$ ("the level") and outputs $2+1$-TQFTs Z_k.

Witten-Reshetikhin-Turaev $2+1$ TQFTs

In this talk we will not recall the full construction of these TQFTs, but we will outline the key ideas for the construction of the vector spaces $Z_k(S_g)$ associated to each surface.
We will follow Witten's clue :

The vector space $Z_k(S_g)$ should be a quantization of the "$SU(2)$-character variety" $\chi_g = \mathrm{hom}(\pi_1(S_g), SU(2))/SU(2)$.

Recall that $\pi_1(S_g) = \langle a_i, b_i, i = 1, \ldots g \rangle / \prod_{i=1}^{g} [a_i, b_i] = 1$.
Therefore a point of χ_g is the datum of $2g$ matrices A_i, B_i in $SU(2)$ which satisfy $\prod_{i=1}^{g} [A_i, B_i] = Id$, up to simultaneous conjugation.
These equations are polynomial in the entries of the matrices thus χ_g is a real algebraic variety.

What is quantisation ?

First of all, a clarification : one “quantizes” a commutative algebra. $\Longrightarrow$ To “quantize a variety” really means to quantize its algebra of functions.

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A rough description of quantization of an algebra is :

1. First "deform" the algebra to get a non commutative one.
2. Then represent this non commutative algebra as operators on some vector space ("the state space").

What is quantisation ?

A prototypical example is that of the Heisenberg Lie algebra generated by operators $P, Q, 1$ such that $\{P, Q\} = \frac{h}{2\pi i}1$.
In quantum mechanics one upgrades the position (Q) and momentum (P) of a particle from commutative functions on $\mathbb{R}$ to elements of a non commutative algebra.
Then one considers these functions (aka "observables") as operators on a Hilbert space (the “state space”).
This is step 1) of our "quantization plan".

What is quantisation ?

To accomplish step 2) (representing the algebra), the following roughly tells us that there is only one choice:

Theorem (Stone-Von Neumann)

Up to isomorphism, there exists a unique irreducible representation of the Lie algebra generated by P and Q via self-adjoint (densely defined) operators on a Hilbert space, and it is the Schrödinger action on $L^2(\mathbb{R})$ given by :

$$Q \cdot f(x) = xf(x), \quad P \cdot f(x) = \frac{h}{2\pi i}\frac{\partial}{\partial x} f(x) \quad \forall f \in L^2(\mathbb{R}).$$

Then the "states" or "wavefunctions" are vectors of this representation, i.e. $f(x) \in L^2(\mathbb{R})$.

What is quantisation ?

Definition (Deformation quantization)

Let $(A, \cdot)$ be a commutative unital algebra. A deformation quantisation of A is the datum of associative products $*_h : A \otimes A \to A \; \forall h \in \mathbb{C}$ such that $*_0 = \cdot$ and for each $f, g \in A$ the following limit exists in A

$$\{f, g\} := \lim_{h \to 0} \frac{f *_h g - g *_h f}{h}$$

Remark (Poisson structure)

Fact: the map $A \otimes A \to A$ given by $\{f, g\}$ is a "Poisson structure" on A (i.e. $\{fg, m\} = f\{g, m\} + g\{f, m\}$, $\{f, g\} = -\{g, f\}$ and $\{\{f, g\}, m\} + \{\{g, m\}, f\} + \{\{m, f\}, g\} = 0$, $\forall f, g, m \in A$).

Example (Quantum plane)

Let $q = exp(h)$. One can quantize the algebra $\mathbb{C}[X, Y]$ via

$$\mathbb{C}_q[X, Y] = \mathbb{C}\{X, Y\}/\langle YX = qXY\rangle.$$

As vector spaces they are isomorphic but the product depends on q. In particular

$$\{Y, X\} = \lim_{h\to 0} \frac{Y *_h X - X *_h Y}{h} = \lim_{h\to 0} \frac{(e^h - 1)X *_h Y}{h} = XY.$$

Recap of the plan to define a $2+1$-TQFT

To define the vector spaces $Z_k(S_g)$ associated to surfaces by Witten's TQFTs, we want to quantise the caracter variety $\chi_g = \mathrm{hom}(\pi_1(S_g), SU(2))/SU(2)$.

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2. Then we have to find a representation of the so-obtained non commutative algebra. The vector space of this representation will be the wanted "quantization of χ_g" and form $Z_k(S_g)$.

If we want to quantize $\chi_g = \hom(\pi_1(S_g), SU(2))/SU(2)$ we need to understand its algebra of functions first.

Definition

Let $\gamma : S^1 \to S_g$ be a curve. Then associated to γ is a function, denoted $tr_\gamma : \chi_g \to \mathbb{C}$, defined as

$$tr_\gamma(\rho) = Trace(\rho(\gamma)), \forall \rho \in \chi_g = \hom(\pi_1(S_g), SU(2))/SU(2).$$

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Theorem (Fricke, Vogt, Procesi, Culler-Shalen, Goldman)

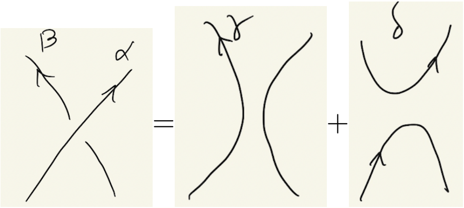
The algebra of regular functions on χ_g is generated by finitely many tr_γ.

A relation satisfied by these functions

If $A, B \in SU(2)$ then $tr(A)tr(B) = tr(AB) + tr(AB^{-1})$.
This directly implies the relation

$$tr_\alpha tr_\beta = tr_\gamma + tr_\delta$$

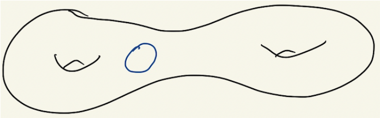
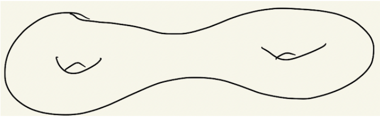
where $\alpha, \beta, \gamma, \delta$ are as in the following drawing:



From now on I won't write tr_α but just depict the curve α to mean the corresponding function on χ_g.

A relation satisfied by these functions

Since $tr(I_2) = 2$, the following is an obvious relation:


$= 2$


i.e. one can drop trivial circles and multiply the remaining function by 2.

Remark

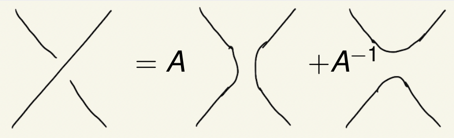
Remark that I no longer orient the curves as the trace is independent on the orientation as $(A) = (A^{-1}) \forall A \in SU(2)$.

Quantizing the character variety: skein algebras

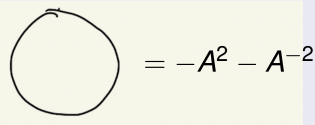
Let now $A = -\exp(h)$ be a quantization parameter. To deform the algebra of functions on $\chi_g = \hom(\pi_1(S_g), SU(2))/SU(2)$, a key idea is to perturb the previous relations to the following "Kauffman relations":

Definition (Skein algebra)

The skein algebra $\mathcal{S}_A(S_g)$ is the associative algebra generated by isotopy classes of closed framed links in $S_g \times [-1,1]$ modulo the following relations:



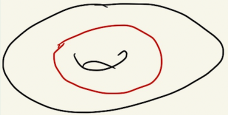
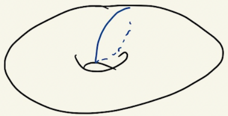
and



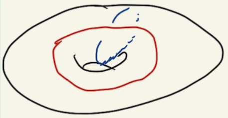
Quantizing the character variety: skein algebras

Example (A product of skeins in the torus)

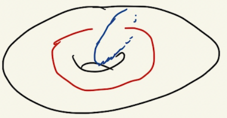
Given curves α, β in the torus :

$\alpha =$  and $\beta =$ 

Then we have that $\alpha\beta$ equals:

 $= A$  $+A^{-1}$ 

While $\beta\alpha$ equals:

 $= A^{-1}$  $+A$ 

Quantizing the character variety: skein algebras

Theorem (Bullock, Frohman, 1997)

$\mathcal{S}_A(S_g)$ is a deformation quantization of the algebra of regular functions on χ_g, with $A = -exp(h)$.
Indeed an isomorphism $iso : \mathcal{S}_{-1}(S_g) \to Reg(\chi_g)$ is given by the map associating to a simple closed curve α the function $-tr_\alpha$.

Furthermore the associated Poisson structure $\{\cdot, \cdot\}$ is Goldman's symplectic form !

Quantizing the character variety: skein algebras

To proceed with the plan of quantizing χ_g now we need to choose a representation of the algebra $\mathcal{S}_A(S_g)$ which will be our state space.
For this let us remark that the Kauffman relations make sense for arbitrary links in an oriented 3-manifold: we only look at local pictures in a 3-ball. Therefore we can define more in general :

Definition (Skein module)

The skein module $\mathcal{S}_A(W^{(3)})$ is the $\mathbb{C}$-vector space generated by framed links in W modulo Kauffman relations.

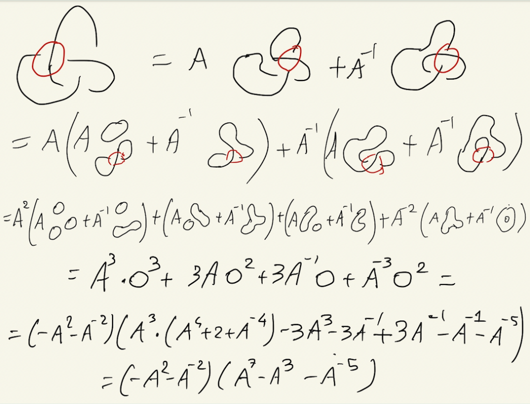
Remark

We no longer have an algebra structure on $\mathcal{S}_A(W^{(3)})$ as there is no way to "superpose" two links in W^3 in general.

Quantizing the character variety: skein algebras

Example

If $W = \mathbb{R}^3$ and L is a knot in W then up to applying the relations to all the crossings of a diagram of L and replacing each of the resulting planar circles by $-A^2 - A^{-2}$ we get a Laurent polynomial in A : it is the Jones polynomial of the knot !



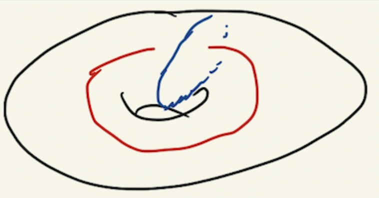
Representations of $\mathcal{S}_A(S_g)$

There is a natural topological way of building representations of $\mathcal{S}_A(S_g)$.

1. Remark that each surface S_g is the boundary of a genus g "handlebody" H_g. And in particular a neighborhood of $\partial H_g = S_g$ is diffeomorphic to $S_g \times [-1, 1]$.
2. Then if $s \in \mathcal{S}_A(S_g)$ and $m \in \mathcal{S}_A(H_g)$ are skeins, one can define the "action of s on m" by simply taking the union skein $s \sqcup m \subset H_g$.

Example

The blue curve is in S_1 while the red curve inside H_1:



Representations of $\mathcal{S}_A(S_g)$

It is easy to verify that this yields a representation:

$$\mathcal{S}_A(S_g) \otimes \mathcal{S}_A(H_g) \to \mathcal{S}_A(H_g).$$

One would like to take $\mathcal{S}_A(H_g)$ as "state space", but it is not finite dimensional, nor a Hilbert space... How to go to finite dimensional vector spaces ?

Skein algebras at roots of unity

If A is "generic" there are no interesting finite dimensional modules over $\mathcal{S}_A(S_g)$. But suppose that $A^{2r} = 1$ (for some $r \geq 3$). Then the following holds :

Theorem (Lickorish, Blanchet-Habegger-Masbaum-Vogel)

For each 3-manifold W, the quotient $\mathcal{S}_A^{red}(W)$ of $\mathcal{S}_A(W)$ by the span of links "colored by r^{th}-Jones Wenzl projectors" is finite dimensional. In particular $\mathcal{S}_A^{red}(H_g)$ is a finite dimensional representation of $\mathcal{S}_A(S_g)$.

The vector spaces $\mathcal{S}_A^{red}(S_g)$ are the sought $Z_r^{WRT}(S_g)$.
(We still have to describe $Z_A^{WRT}(W)$ for each cobordism W, but we will not continue in this direction...)

Not the end of the story...

All this happened before 1997.
I would like to highlight some of the evolutions of this story...
Most statements are going to be imprecise, they aim just at giving an idea of the current evolution of the research and to connect with the topics of the Skein Days.

In our account we started from the algebra $\mathcal{S}_A(S_g)$ and chose one specific finite dimensional representation : the action on $\mathcal{S}_A^{red}(H_g)$.
This was giving us our "state space" associated to S_g by the TQFT.
But it turns out there are many other possible finite dimensional representations of $\mathcal{S}_A(S_g)$...

Bonahon-Wong's classical shadows

Theorem (Bonahon-Wong, 2016)

Let A be a root of unity of order congruent to $2 \mod 4$. Then the center Z of $\mathcal{S}_A(S_g)$ contains the (commutative) algebra of regular functions on χ_g and $\mathcal{S}_A(S_g)$ is finitely generated over Z.
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As a consequence, if V is an irreducible finite dimensional module over $\mathcal{S}_A(S_g)$ then there exists a point $\rho = \rho(V) \in \chi_g$ (the "classical shadow") such that

$$z \cdot v = z(\rho)v, \ \forall z \in Z = Fun(\chi_g), \forall v \in V.$$

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And it turns out that for $V = \mathcal{S}^{red}(H_g)$ the point is $\rho = trivial$ (i.e $\rho : \pi_1(S_g) \to SU(2)$ sends every element to Id).

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And it turns out that for $V = \mathcal{S}^{red}(H_g)$ the point is $\rho = trivial$ (i.e $\rho : \pi_1(S_g) \to SU(2)$ sends every element to Id). **So there's a full variety of other potential "quantum states" to explore !**

Non semi-simple TQFTs

In a joint work with C. Blanchet, N. Geer and B. Patureau we constructed TQFTs whose classical shadows correspond to the $\rho : \pi_1(S_g) \to SL(2, \mathbb{C})$ taking value in the diagonal matrices. These are nothing else than cohomology classes on S_g...

Theorem (BCGP 2014, vague statement)

There exists a TQFT for surfaces and cobordisms endowed with cohomology classes. This TQFT is non semi-simple, namely the action of some elements of the mapping class groups of surfaces contains non diagonal Jordan blocks.

The non-semi simplicity corresponds to the fact that, on the algebraic level, the theory uses a category which is not semi-simplified and contains genuine projective objects.

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Conjecture

These TQFTs provide finite dimensional faithful representations of the Torelli groups (which are unknown to be linear).

Skein modules as TQFTs ?

Remark

We associated to each 3-manifold W a vector space $\mathcal{S}_A(W)$ but then we produced "only" a $2+1$-TQFT. Can't we produce a $3+1$ TQFT ? For this to be possible the vector spaces should be finite dimensional for closed manifolds...

Theorem (Gunningham-Jordan-Safronov, 2020)

Let A be a formal parameter and work over $\mathbb{Q}(A)$. Let $\partial W = \emptyset$ and W be compact. Then $\dim_{\mathbb{Q}(A)} \mathcal{S}_A(W) < \infty$.

The proof of this striking result combines techniques from factorisation homology with the theory of holonomic systems of differential equations.

Skein modules as TQFTs ?

Still, there is no general formula to compute the dimension but the following important result gives a clear idea:

Theorem (Detcherry-Kalfagianni-Sikora, 2023 (imprecise statement))

If W is sufficiently "tame" (the skein module $\mathcal{S}_A(M)$ has no torsion of a certain type and the algebra of functions on the variety $\chi_g(W) = \hom(\pi_1(W), SL(2))^{SL(2)}$ has dimension $d < \infty$) then $\dim_{\mathbb{Q}(A)}(\mathcal{S}_A(W)) = d$.

Question (Witten-Kapustin)

Can one obtain a $3+1$ TQFT whose state spaces are $\mathcal{S}_A(W)$?

More general TQFTs

It turns out that the previous “skein rules” can be generalised very much.
Indeed one can define “skein modules” and “skein algebras” and even a “skein category” associated to any so-called “ribbon category $\mathcal{C}$”. (See for instance Jennifer Brown’s talk).
What we saw corresponds to $\mathcal{C} =$ the category of modules over the so-called quantum group $U_A(\mathfrak{sl}_2)$: secretely every link was decorated by the natural 2-dimensional module over $\mathfrak{sl}_2$.
If the category is “sufficiently small and non degenerate” (“modular”) then there is an associated “$2+1$-Reshetikhin-Turaev TQFT.”
These TQFTs are “semi-simple”, i.e. the vector spaces $Z(S_g)$ are semi-simple modules over $MCG(S_g)$.

Some TQFTs $3+1$ from skein spaces

In collaboration with B. Haioun, N. Geer and B. Patureau we proved:

Theorem (CGHP 2023, (imprecise statement))

For each category C with "chromatic morphism" and "modified trace" there exists a $3+1$ TQFT whose state spaces are of the form $\mathcal{S}_C(W)$.

This is not yet an answer to the previous question but it goes in that direction...

Thank you for your attention !

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