# NOTES OF THE TALK OF L. TAELMAN: LOCAL SYSTEMS AND HIGGS BUNDLES

(NOTES TYPED BY A. SAUVAGET)

## 1. Introduction: from local systems to vector bundles

$X$ Riemann surface of $g(X) \geq 2$.

**Definition 1.1.** A *local system* on $X$ is a locally constant sheaf of $\mathbb{C}$-vct spaces, with finite dimensional stalks

Equivalence: $\{$local systems$\} \leftrightarrow \{$ representation of $\pi_1(X,x)\}$:

$$\mathbb{E} \mapsto \mathbb{E}_x.$$

The key player: $\{$local systems$\} \to \{$ vector bundles on $X\}$,

$$\mathbb{E} \mapsto \mathcal{E} = \mathbb{E}_{\mathbb{C}} \otimes \mathcal{O}_X.$$

It is an exact functor.

**Example 1.2.** local systems of rank 1: $H^1(X, \mathbb{C}^\star) = \mathrm{Hom}(\pi_1, \mathbb{C}^\star) \simeq (\mathbb{C}^*)^{2g}$
vector bundles of rk $1 = H^1(X, \mathcal{O}_X^*) = \mathrm{Pic}(X)$ (these two are algebraic varieties).

Then we have two exact sequences with a natural morphism between them:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}_X \arrow[r] \arrow[d, "\simeq"'] & \mathbb{C}_X \arrow[r, "\exp"] \arrow[d] & \mathbb{C}_X^\star \arrow[r] \arrow[d] & 1 \\
0 \arrow[r] & \mathbb{Z}_X \arrow[r] & \mathcal{O}_X \arrow[r, "\exp"] & \mathcal{O}_X^\star \arrow[r] & 1
\end{tikzcd}$$

And we have the exact sequence:

$$0 \to H^0(X, \Omega_X^1) \to H^1(X, \mathbb{C}_X) \to H^1(X, \mathcal{O}_X) \to 0$$

from this, we get:

$$0 \to H^0(X, \Omega^1) \to H^1(X, \mathbb{C}^*) \to H^1(X, \mathcal{O}_X^*) \to \mathbb{Z} \to 0$$

(the second is $(\mathbb{C}^*)^{2g}$ and the third $\mathrm{Pic}(X)$)

## 2. Unitary local systems

**Definition 2.1.** A local system $\mathbb{E}$ is *unitary* if there exists an hermitian positive definite form $h : \mathbb{E} \times \mathbb{E} \to \mathbb{R}_X$. Equivalently $\mathbb{E}$ comes from $\rho : \pi_1(X,x) \to U(n)$.

**Proposition 2.2.** *$\{$unitary local systems$\} \to \{$ vector bundles on $X\}$ is fully faithful.*

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*Date*: 31/01/2020.

*Proof.* claim: $\mathbb{E}$ unitary $H^0(X,\mathbb{E}) \simeq H^0(X,\mathcal{E})$ is an isomorphism (this claim implies the proposition as take $\mathbb{E} = \underline{\mathrm{Hom}}(\mathbb{E}_0, \mathbb{E}_1) = \mathbb{E}_0^\vee \otimes_1 z \mapsto H())$

Indeed, first remark that

$$f \in H^0(X,\mathcal{E}) = \{f : \tilde{X} \to \mathbb{E}_x, \text{s.t. } f \text{ is holomorphic and } \pi_1\text{-equiv}\}$$

(where $\widetilde{X}$ is the universal cover).We consider the map:

$$f \mapsto ||f||(\widetilde{X} \to \mathbb{E}_x \to \mathbb{R})$$

this function is sub-harmonic and thus constant on $\widetilde{X}$. Thus $f$ comes from a section of the local system. ☐

Then what is the essential image of this functor?

## 3. Moduli space of (stable) vector bundles

We fix numerical invariants: $n = rank(\mathcal{E}) = c_0$ and $d = deg(\mathcal{E}) = c_1$.

Assume that $\mathcal{M}(n,d)$ is the moduli space of rank $n$ and degree $d$ vector bundles on X:

(1) not fine, because Aut($\mathcal{E}$) non-trivial: $\mathcal{E} \to X \times T$ and we consider: $\mathcal{E} \otimes \pi_T^* \mathcal{L} \to X \times T$ have isomorphic fibers over $X \times \{t\}$.
(2) not seperated.
(3) not bounded.

**Definition 3.1.** The slope of $\mathcal{E} = deg(\mathcal{E})/rk(\mathcal{E}) = \mu(\mathcal{E})$.

$\mathcal{E}$ is semi-stable if for any strict non-trivial sub-sheaf $\mathcal{F}$ we have $\mu(\mathcal{F}) \leq \mu(\mathcal{E})$

$\mathcal{E}$ is stable if for any strict non-trivial sub-sheaf $\mathcal{F}$ we have $\mu(\mathcal{F}) < \mu(\mathcal{E})$

(1) line bundles are stable
(2) $\mathcal{E}, \mathcal{F}$ semi-stable, $\mu(\mathcal{E}) > \mu(\mathcal{F})$, then $\mathrm{Hom}(\mathcal{E},\mathcal{F}) = 0$.
(3) $\mathcal{E}, \mathcal{F}$ stable and $\mu(\mathcal{E}) = \mu(\mathcal{F})$ then either $\mathcal{E} \simeq \mathcal{F}$ or $\mathrm{Hom}(\mathcal{E},\mathcal{F}) = 0$.
(4) $\mathcal{E}$ stable, then $End(\mathcal{E}) = \mathbb{C}$, $\mathcal{E}$ is simple.

So we need more constraints on this moduli space to have good modular properties.

**Theorem 3.2.** *(HN filtration) $\forall \mathcal{F}$, there exists a unique filtration*

$$0 = \mathcal{E}_0 \subset \ldots \subset \mathcal{E}_k = \mathcal{E}$$

*s.t. $\mathcal{E}_i/E_{i-1}$ is semi-stable, $\mu(\mathcal{E}_{i+1}/\mathcal{E}_i)$,*

*(JN filtration) $\forall$ semistable $\mathcal{E}$ there exists*

$$0 = \mathcal{E}_0 \subset \ldots \subset \mathcal{E}_k = \mathcal{E}$$

*s.t. $\mathcal{E}_{i+1}/\mathcal{E}_i)$ are stable of slope $\mu(\mathcal{E})$.*

**Definition 3.3.** $\mathcal{E}$ is polystable if $\mathcal{E} = \bigoplus \mathcal{E}_i$, all of the same slope.

**Theorem 3.4.** *(Mumford, Seshadri) $\exists$ and irreducible, projective, variety $\mathcal{M}^{ss}(n,d)$, containing an open $\mathcal{M}^s(n,d)$ smooth s.t.*

*(1) $\mathcal{M}^s(n,d)$ is the coarse moduli space for stable vector bundles of rank $n$ and deg $d$.*
*(2) if $gcd(n,d) = 1$, then $\mathcal{M}^s = \mathcal{M}^{ss}$, they represent a functor:*

$$T \mapsto \{\mathcal{E} \to X \times T | \textit{the fibers over any } t \in T \textit{ is a semi-stable v.b. on } X\}/\mathrm{Pic}^0(T)$$

*(3) $\mathcal{M}^{ss}(n,d) \leftrightarrow$ polystable vect bundles of rank $n$ and deg $d$.*

## 4. Narasimhan-Seshadri theorem

**Theorem 4.1.** *(NS)* $\mathbb{E} \mapsto \mathcal{E} = \mathbb{E} \otimes \mathcal{O}_X$ *defines an equivalence of category: unitary local systems / polystable vector bundles of slope 0.*

*Sketch of the proof.* The steps of the proof:

(1) $\mathbb{E}$ irreducible unitary, then $\mathcal{E}$ is stable of slop 0.
   Trick of this step: if $\mathcal{F} \subset \mathcal{E}$, then consider det$\mathcal{F} \subset \bigwedge^{rk(\mathcal{F})} \mathcal{E}$ which is the exterior power of a local system (then you work a little bit more...)

(2) fully faithful: has already been stated.

(3) essentially surjective?
   $\mathbb{E}$ unitary, then $\mathcal{E}$ is polystable and $\mathcal{E}$ stable $\Leftrightarrow$ $\mathbb{E}$ irreducible. I.e. we have a cartesian diagram:

$$\begin{tikzcd} \mathrm{Hom}(\pi_1, U(n)) \arrow[r, "\psi"] & \mathcal{M}^{ss}(n,0) \\ \emptyset \neq \mathrm{Irr}(\pi_1, U(n)) \arrow[u] \arrow[r, "\psi_0"'] & \mathcal{M}^{s}(n,0) \arrow[u] \end{tikzcd}$$

   Facts: diagram is cartesian and $\psi_0$ is submersion of smooth manifolds (deformation theory).
   $\Rightarrow$: the statement. Indeed $\mathrm{Hom}(\pi_1, U(n))$ is compact, and $\psi_0$ is proper and thus the image of $\psi_0$ is closed (as $\psi_0$ is a submersion). Besides, $\mathcal{M}^s(n,0)$ is irreducible, so the image of $\psi_0$ is the total space.

## 5. Higgs bundles

**Definition 5.1.** A Higgs bundle on $X$ is $(\mathcal{E}, \theta)$ with $\mathcal{E}$ vect bundle $\theta : \mathcal{E} \xrightarrow{\theta} \mathcal{E} \otimes \Omega_X$ which is $\mathcal{O}_X$ linear.

We say that a Higgs bundle is stable if for all sub-Higgs bundle, then $\mu(\mathcal{F}) < \mu(\mathcal{E})$ (and polystability is also defined as earlier).

**Theorem 5.2.** $\exists$ *an equivalence of categories:* $\{$ *local systems*$\} \to \{$ *to polystable Higgs bundles of slope* $0\}$.

*It is compatible with the above equivalence of categories if we map* $\{$*unitary local sytems* $\} \to \{$*polystable vect bundles of slope 0*$\}$ *by* $\mathcal{E} \to (\mathcal{E}, 0)$.

Why is this called non-abelian Hodge theory? a possible explanation:
$\mathbb{G}_a$- Hodge theroy: $H^1(X, \mathbb{C}) = H^1(X, \mathcal{O}_X) \times H^0(X, \Omega^1_X)$
$\mathbb{G}_m$- Hodge theroy: $H^1(X, \mathbb{C}^\star) = \mathrm{Pic}^0(X) \times H^0(X, \Omega^1_X)$
$GL_n$- Hodge theroy: $H^1(X, GL_n) \simeq$ polystable ... NAHT.

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