PERVERSE SHEAVES

BY LIE FU (NOTES BY A. SAUVAGET)

1. T-STRUCTURES

$\mathcal{A}$= abelian category $\rightsquigarrow D^b(\mathcal{A})$: triangulated category.

Question: How to recover $\mathcal{A}$ inside $D^b(\mathcal{A})$?

Truncation: $E \in D^b(\mathcal{A})$ and $n \in \mathbb{Z}$), we denote:

$$\tau^{\leq n}(E) = (\ldots \to E^{n-1} \to ker(E^n \to E^{n+1}) \to 0)$$

similarly for $\tau^{\geq n}(E)$.

Then we have a distinguished triangle: $\tau^{\leq n}(E) \to E \to \tau^{\geq n+1}(E) \to +1$, and

$$\mathcal{A} \simeq \{E | H^i(E) = 0, \forall i \neq 0\}$$

Definition 1.1. If D is a triangulated category, a *t-structure* on D is a pair of full additive subcategories $(D^{\leq 0}, D^{\geq 0})$. We denote by $D^{\leq n} = D^{\leq 0}[-n]$ and $D^{\geq n} = D^{\geq 0}[-n]$. We impose the conditions:

(1) $D^{\geq -1} \subset D^{\geq 9} \subset D^{\geq 1} \ldots$
(2) (Semi-orthogonality) $\mathrm{Hom}(D^{\leq 0}, D^{\geq 1}) = 0$.
(3) (decomposability) $\forall E \in D$, $\exists$ distinguished triangle:

$$E' \to E \to E'' \to +1$$

with $E' \in D^{\leq 0}$ and $E'' \in D^{\geq 1}$

The classical situation provides an example of t-structure on $D(\mathcal{A})$.

Definition 1.2. The category $\mathcal{C} = D^{\leq 0} \cap D^{\geq 0}$ is called the *heart* of the t-structure.

Proposition 1.3. *The decomposition of objects is canonical.*

Proof. If we have two distinct decompositions of E: (E', E'') and (F', F''). Then we have a long exact sequence:

$$\ldots \to \mathrm{Hom}(E', F''[-1]) \to \mathrm{Hom}(E', F') \to \mathrm{Hom}(E', E) \to \mathrm{Hom}(E', F'') \to \ldots$$

The two exterior are zero by semi-orthogonality condition. □

Corollary 1.4. *For all $n \in \mathbb{Z}$, we have functors: $\tau^{\leq n} : D \to D^{\leq n}$ and $\tau^{\geq n} : D \to D^{\geq n}$, s.t. $(i_{\leq n}, \tau^{\leq n})$ and$(\tau^{\geq n}, i_{\geq n})$ are adjoint pairs.*

Remark 1.5. $\mathrm{Hom}(D^{\leq n}, D^{\geq n+1}) = 0$ by definition. We have actually: $D^{\leq n} =$ orthogonal of $D^{\geq n+1}$.

Theorem 1.6. *(BDD) $\mathcal{C} = D^{\leq 0} \cap D^{\geq 0}$ is an abelian category, and all short exact sequences $0 \to E' \to E \to E'' \to 0$ in $\mathcal{C}$ give rise to a distiguinshed triangle $E' \to E \to E'' \to +1$ in D.*

Proof.
- $\mathcal{C}$ is closed under $\oplus$.

Date: 31/01/2020.

- let $E, F \in \mathcal{C}$, $f : E \to F$ complete into $E \to F \to G \to +1$ distinguished. Then $ker(f) = H^{-1}(G) = \tau^{\leq -1}\tau^{\geq -1}(G)$, and $coker(f) = H^0(G) = \tau^{\leq 0}\tau^{\geq 0}(G)$.
  $D^{\leq n}$ is stable by extension. Then for all W in $\mathcal{C}$ we have to show that
  $$0 \to \mathrm{Hom}(W, ker(f)) \to \mathrm{Hom}(W, kerE) \to \mathrm{Hom}(W, F)$$
  is exact. We use the above distinguished triangle to see that:
  $$\mathrm{Hom}(W, F[-1]) = 0 \to \mathrm{Hom}(W, G[-1]) \to \mathrm{Hom}(W, E) \to \mathrm{Hom}(W, F)$$
  The first term is $\mathrm{Hom}(W, ker)$. (same for coker, while the proof for images and co-images uses the octahedron axiom

□

We can in fact defined truncation functors: $\tau^{[a,b]} : D \to D^{\geq a} \cap D^{\leq b}$.

Proposition 1.7. *$H^0 = \tau^{\leq 0} \circ \tau^{\geq 0}$ is a cohomological functor from D to $\mathcal{C}$, i.e. for any distinguished triangle, we have a long exact sequence.*

2. Exactness

Let $F : D_1 \to D_2$ be a triangulated functor between triangulated endowed with t-structures. (giving $\mathcal{C}_1$ and $\mathcal{C}_2$).

Definition 2.1. ${}^pF : \mathcal{C}_1 \to D_1 \to D_2 \to \mathcal{C}_2$

Definition 2.2. F is left t-exact if $F(D^{\geq 0}) \subset D_2^{\geq 0}$ (and right...)

Proposition 2.3. *If F is left t-exact then ${}^pF$ is left exact (same for right).*

Proposition 2.4. *$F : D_1 \leftrightarrow D_2 : G$ adjoint pair, then F right t-exact is equivalent to G left t-exact.*

3. Perverse t-structure

X:analytic space (algebraic variety/ $\mathbb{C}$).

$D^b(\mathbb{C}_X)$ = bounded derived cat of $\mathbb{C}_X$-modules.

$D^n_c(\mathbb{C}_X) = D^b_c(X)$ = the category of $E \in D^b(\mathbb{C}_X)$, such that for all i, $H^i(E) \in Sh(X)$ is constructible.

Recall a sheaf F on X is constructible (algebraically) if there exists an alg. stratification of X such that F is a local system on each stratum.

Six functor formalism for $f : X \to Y$:

- $f^* : D^b(\mathbb{C}_Y) \to D^b(\mathbb{C}_X)$;
- $f_* = Rf_* : D^b(\mathbb{C}_X) \to D^b(\mathbb{C}_Y)$;
- $f_! : D^b(\mathbb{C}_X) \to D^b(\mathbb{C}_Y)$;
- $f^! : D^b(\mathbb{C}_Y) \to D^b(\mathbb{C}_X)$;
- $\otimes$ and $RHom(,)$.

Adjoint pairs (f^*, f_*) and $(f_!, f^!)$.

Verdier duality: $\pi : X \to pt$, $\omega_X = \pi^!(\mathbb{C})$.

$$\mathbb{D}_X = (E \mapsto RHom(E, \omega_X))$$

Proposition 3.1. *(1) $\mathbb{D}_X^2 \simeq Id_{D^b_c(X)}$*
(2) $f_* \circ \mathbb{D}_X = \mathbb{D}_Y \circ f_!$
(3) $f^! \circ \mathbb{D}_Y = \mathbb{D}_X \circ f^*$ on D^b_c.

Definition 3.2. (Perverse t-structure) F in ${}^pD^{\leq 0}_C(X)$ if $\dim(\mathrm{supp}(H^i(F)) \leq -i$ for all i.

F in ${}^pD^{\geq 0}_C(X)$ if $\dim(\mathrm{supp}(H^i(DD_X(F))) \leq -i$ for all i.

$Perv(X)$ is the heart of this t-structure.

Theorem 3.3. *(BBD) This is a t-structure.*

The proof goes by constructing the t-structure obtained by gluing pieces.

Definition 3.4. A *recollement* is the datum of: $D_Z \overset{i_*}{\to} D \overset{j_*}{\to} D_U$ that gives two triples of adjoints $(i^*, i_*, i^!)$, $(j_!, j^*, j_*)$ such that:

(1) $j^* \circ i_* = 0$
(2) $j_! j^* \to id \to i_* i^*$ distinguished
(3) $i_* i^! \to id \to j_* j^*$distinguished
(4) $i_*, j_!, j_*$ are fully faithfull.

Claim: From t-structures on D_Z and D_U, then $D^{\leq 0} = \{E / i^* E \in D^{\leq 0}, j^* E \in D^{\leq 0}_U\}$ is a t-structure on D.